Intermediate
Mathematics League
of
Eastern Massachusetts

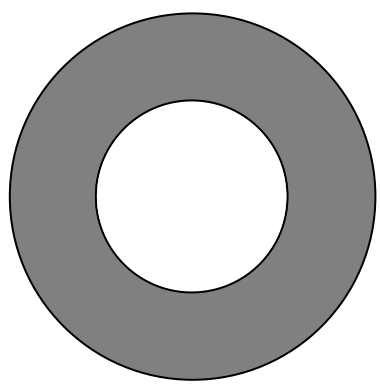
Meet #1
October 2005

Category 1
Mystery
Meet #1, October 2005

1. There are five kids in the Braeburn family. They are all spaced three years apart, and the oldest is three times as old as the youngest. How old is the middle child in the Braeburn family?

2. Shayna has 24 coins consisting of quarters and dimes. If her quarters were dimes and her dimes were quarters, her coins would be worth 90 cents more. How much are her coins worth?

3. What is the greatest number of regions the shaded ring below can be divided into using three straight lines?



Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 1
Mystery
Meet #1, October 2005

Answers

1. 12

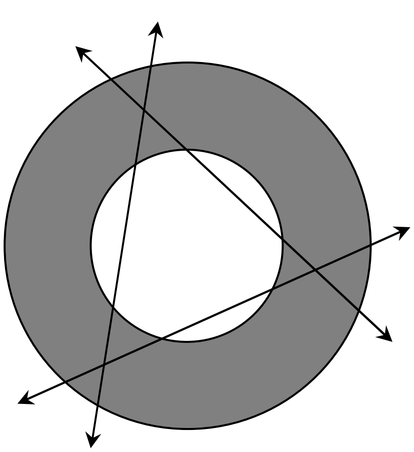
2. \$3.75 or 375¢

3. 9

1. Suppose the youngest child is a years old. Then the other four children are $a + 3$, $a + 6$, $a + 9$, and $a + 12$ years old. If the oldest is three times as old as the youngest, then $a + 12 = 3a$. The extra 12 years must account for the extra $2a$, so $a = 6$ years. The middle child in the Braeburn family must be 6 + 6 = **12** years old.

2. If Shayna had the same number of each coin, then her coins would be have the same value if she swapped quarters for dimes and dimes for quarters. Since she would gain 90 cents by swapping, we know that she must have more dimes than quarters. Every extra dime that is swapped for a quarter yields 15 cents. Since $90 \div 15 = 6$, Shayna must have 6 more dimes than quarters. The remaining $24 - 6 = 18$ coins must be 9 quarters and 9 dimes. In all, Shayna has $6 + 9 = 15$ dimes and 9 quarters, worth $15 \times 10 + 9 \times 25 = 150 + 225 = 375$¢ or \$3.75.

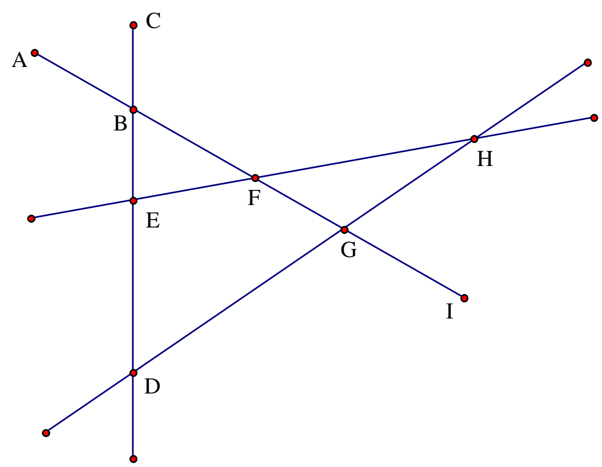
3. The ring can be divided into **9** regions as shown in the diagram below.



Category 2
Geometry
Meet #1, October 2005

1. Tim added x degrees to a 27-degree angle. The complement of this new angle was 48 degrees. He then added y degrees to this 48-degree angle. The complement of this new angle was 7 degrees. Find the value of $x + y$.

2. In the figure at right, the measure of angle ABC is 60 degrees, the measure of angle DEF is 100 degrees, and the measure of angle DGI is 116 degrees. How many degrees are in the measure of angle FHG?



3. The sum of the complement of angle x and the supplement of angle x is 10 degrees less than eight times the angle x. How many degrees are in the measure of angle x?

Answers
1. ______________
2. ______________
3. ______________

Solutions to Category 2
Geometry
Meet #1, October 2005

Answers

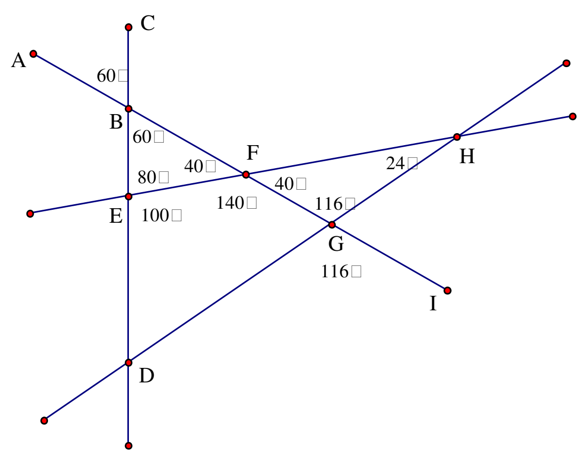
1. 50

2. 24

3. 28

1. The measures of two complementary angles add up to 90 degrees. Since 27 + 48 = 75, the unknown amount x must have been 90 – 75 = 15 degrees. Likewise, since 48 + 7 = 55, y must be 90 – 55 = 35. The value of $x + y$ is thus 15 + 35 = **50**.

2. Vertical angles are congruent, straight angles have a sum of 180 degrees, and triangles have an angle sum of 180 degrees. Using these three facts, the angles of the two small triangular regions can be determined from the angle measures given. The measure of angle FHG is **24** degrees.



3. The complement of angle x is 90 – x and the supplement of angle x is 180 – x. Their sum is (90 – x) + (180 – x) = 270 – 2x. We know that this sum is equal to ten less than eight times angle x, or 8x – 10. Now we can write an equation and solve for x.

$$270 - 2x = 8x - 10$$
$$+10 \quad = \quad +10$$
$$280 - 2x = 8x$$
$$+2x = +2x$$
$$280 = 10x$$
$$x = 28$$

Category 3
Number Theory
Meet #1, October 2005

1. Two prime numbers, each less than 50, have a difference of 35. What is the sum of these two primes?

2. The letters A and B each represent one digit in the five-digit number $AB9BA$. If $AB9BA$ is divisible by 2, 3, 4, 6, 8, and 9, what is the value of $2A + B$?

3. How many whole numbers between 80 and 100 are the product of two distinct primes? In other words, how many whole numbers between 80 and 100 have exactly four factors?

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 3
Number Theory
Meet #1, October 2005

Answers

1. 39

2. 15

3. 8

1. After 2, all primes are odd. Any two odd primes will have an even difference. If the difference of two primes is odd, then 2 must be one of the primes. Since our difference is 35, the other prime must be 35 + 2 = 37. Our two primes are 2 and 37 and their sum is 2 + 37 = **39**.

2. The sum $A + B + 9 + B + A$ must be a multiple of 9, since the five-digit number $AB9BA$ is divisible by 9. This means that $A + B = 9$. Also, A must be even, since $AB9BA$ is a divisible by 2. The possibilities for the last two digits BA are 18, 36, 54, 72, and 90. Only 36 and 72 are divisible by 4, so the 5-digit number is either 63936 or 27972. Since 936 is divisible by 8 and 972 is not, the number must be 63936. So $A = 6$ and $B = 3$, and the value of $2A + B$ is $2 \times 6 + 3 =$ **15**.

3. The prime factorizations of all the numbers between 80 and 100 are listed below. The **8** that are products of two primes are in bold.

$81 = 3^4$
$\mathbf{82 = 2 \cdot 41}$
$83 = 83$
$84 = 2^2 \cdot 3 \cdot 7$
$\mathbf{85 = 5 \cdot 17}$
$\mathbf{86 = 2 \cdot 43}$
$\mathbf{87 = 3 \cdot 29}$
$88 = 2^3 \cdot 11$
$89 = 89$
$90 = 2 \cdot 3^2 \cdot 5$

$\mathbf{91 = 7 \cdot 13}$
$92 = 2^2 \cdot 23$
$\mathbf{93 = 3 \cdot 31}$
$\mathbf{94 = 2 \cdot 47}$
$\mathbf{95 = 5 \cdot 19}$
$96 = 2^5 \cdot 3$
$97 = 97$
$98 = 2 \cdot 7^2$
$99 = 3^2 \cdot 11$

Category 4
Arithmetic
Meet #1, October 2005

1. Find the value of the expression below.

$$10 \cdot 3^2 - \frac{5^2 - 11}{7} + 18 \div 3 \cdot 2$$

2. Each X in the lineplot below shows the score of a mathlete on a certain middle school math team. Find the median and the mode of these scores?

				X					
				X	X		X		
			X	X	X		X		
			X	X	X	X	X		X
	X		X	X	X	X	X	X	X
0	2	4	6	8	10	12	14	16	18

3. So far this quarter, Gil has taken 6 quizzes and 2 tests, and there is one more test to take. His scores on the quizzes are 85, 83, 98, 93, 86, and 87. His scores on the tests are 91 and 87. In computing an average, Gil's teacher counts each test as two quizzes. He will also round an average of 89.5 up to a 90, which is an A-. What is the lowest score that Gil can get on the final test of the quarter and still have an average of 89.5?

Answers

1. _______________

2. Median _________

   Mode __________

3. _______________

Solutions to Category 4
Arithmetic
Meet #1, October 2005

Answers

1. 100

2. Median 10
Mode 8

3. 93

1. Following the order of operations, we get

$$10 \cdot 3^2 - \frac{5^2 - 11}{7} + 18 \div 3 \cdot 2$$
$$= 10 \cdot 9 - \frac{25 - 11}{7} + 6 \cdot 2$$
$$= 90 - \frac{14}{7} + 12$$
$$= 90 - 2 + 12$$
$$= \mathbf{100}$$

2. The median score is the middle score when the scores are arranged from least to greatest. The middle of these 22 scores is the average of the eleventh and twelfth scores. Since these are both 10, the median is **10** points. The mode is the most frequent score, which is **8** points, made obvious by the tallest bar of X's. (Both answers must be correct for the 2 points to be awarded.)

3. The three tests that Gil will take in the quarter count as much as six quizzes, since each one counts double. If Gil is to have an average of 89.5, he must have a total of 89.5 × 12 = 1074 points. So far he has 85 + 83 + 98 + 93 + 86 + 87 = 532 points in quizzes and 2 × 91 + 2 × 87 = 182 + 174 = 356 points in tests, for a total of 888 points. He needs 1074 – 888 = 186 points. Dividing this by 2, we see that Gil must get at least a **93** on his last test to earn an A-.

Category 5
Algebra
Meet #1, October 2005

1. Evaluate the expression below for x = -13 and y = -7.

$$(x+y)(x-y)-(x^2-y^2)$$

2. Find the value of B that will make the equation below an identity.
(An identity is an equation for which all real numbers are solutions.)

$$17(3x-5)-(9x-1)=B(3x-6)$$

3. Solve for x.

$$\frac{8\left(\frac{5(3x+12)}{22}-7\right)-1}{3}+4=25$$

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 5
Algebra
Meet #1, October 2005

Answers

1. 0

2. 14

3. 18

1. The expression is equal to zero regardless of the values of *x* and *y*. Expanding the product $(x+y)(x-y)$, we get $x^2 - xy + xy - y^2$, which reduces to $(x^2 - y^2)$. Substituting this into the original expression, we see that $(x^2 - y^2) - (x^2 - y^2) = 0$. Using the particular values given for *x* and *y*, the expression can be evaluated as follows:

$$((-13)+(-7))((-13)-(-7)) - ((-13)^2 - (-7)^2)$$
$$= (-20)(-6) - (169 - 49)$$
$$= 120 - 120 = \mathbf{0}$$

2. First, let's simplify both sides of the equation.

$$17(3x-5) - (9x-1) = B(3x-6)$$
$$51x - 85 - 9x + 1 = 3Bx - 6B.$$
$$42x - 84 = 3Bx - 6B$$

If we are to have an identity, then $3Bx$ must equal $42x$ and $-6B$ must equal -84. This will happen when B = **14**.

3. We need to work backwards, doing the inverse operation at each stage. Start with 25, subtract 4, multiply by 3, add 1, divide by 8, add 7, multiply by 22, divide by 5, subtract 12, and finally divide by 3. The result is that x = **18**.

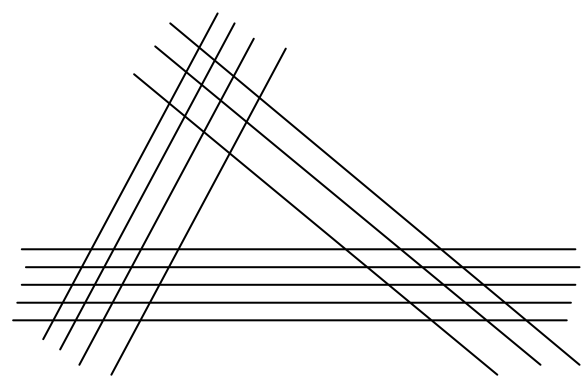
Category 6
Team Questions
Meet #1, October 2005

1. To multiply a certain three-digit number by 83, you can just put a 2 in front of the number and a 2 at the end. What is the three-digit number?

2. Two opposite sides of a square garden are increased by 8 feet. The other two opposite sides are decreased by 8 feet. By how many square feet is the area of the garden changed? (Note: A positive number would indicate an increase in area and a negative number would indicate a decrease in area.)

3. What is the sum of all the positive factors of 1994?

4. How many triangles are there in the figure at right?



5. What is the sum of all integer values of n such that $\frac{24}{2n+1}$ is an integer?

Answers

1. ______________ $= A$
2. ______________ $= B$
3. ______________ $= C$
4. ______________ $= D$
5. ______________ $= E$
6. ________________

6. Using the values the team obtained in questions 1 through 5, evaluate the following expression:

$$D \div \sqrt{\frac{-C}{\sqrt[3]{B}+E} - A}$$

Solutions to Category 6
Team Questions
Meet #1, October 2005

Answers

1. 274

2. –64

3. 2994

4. 60

5. -2

6. 4

1. Let's call the three-digit number x. Then $83x = 20002 + 10x$. Subtracting $10x$ from both sides of the equation, we get $73x = 20002$. Dividing 20002 by 73, we find that $x =$ **274**.

2. Let's say the original side length of the square garden is x feet. Then the area of the square garden is x^2 feet. The length and width of the new rectangular garden are then $x + 8$ and $x - 8$. The area of the new garden is $(x+8)(x-8) = x^2 - 8x + 8x - 64 = x^2 - 64$ square feet. The area has thus decreased by 64 square feet, which can be written as **–64**.

3. The prime factorization of 1994 is 2×997. To determine that 997 is prime, it is only necessary to verify that the primes less than the square root of 997 do not divide it. Since $\sqrt{997} \approx 31.57$, we must check for divisibility by 2, 3, 5, 7, 11, 13, 17 19, 23, 29 and 31. None of them divide 997. The factors of 1994 are thus 1, 2, 997, and 1994. Their sum is **2994**.

4. Any triangle in the figure must use one of the 4 lines on the left, one of the 3 lines on the right, and one of the 5 lines on the bottom. There are $4 \times 3 \times 5 = 60$ ways to pick one line from each set, so there must be **60** triangles in the figure.

5. If the fraction $\frac{24}{2n+1}$ is to be an integer, then 2*n* + 1 must be a factor of 24. The factors of 24 are 1, 2, 3, 4, 6, 8, 12, 24, *and their negatives*. We now let 2*n* + 1 equal each of these factors and solve for *n*. It becomes apparent that only the odd factors of 24 will result in integer values for *n*. The four equations for which *n* is an integer are 2*n* + 1 = -3, 2*n* + 1 = -1, 2*n* + 1 = 1, and 2*n* + 1 = 3. Their solutions are *n* = -2, *n* = -1, *n* = 0, and *n* = 1, respectively. The sum of these values is **–2**.

6. Substituting the appropriate values into the expression and simplifying, we get:

$$
\begin{aligned}
D \div \sqrt{\frac{-C}{\sqrt[3]{B}+E} - A} &= 60 \div \sqrt{\frac{-2994}{\sqrt[3]{-64}+(-2)} - 274} \\
&= 60 \div \sqrt{\frac{-2994}{-4+(-2)} - 274} \\
&= 60 \div \sqrt{\frac{-2994}{-6} - 274} \\
&= 60 \div \sqrt{499 - 274} \\
&= 60 \div \sqrt{225} \\
&= 60 \div 15 \\
&= \mathbf{4}
\end{aligned}
$$