Meet #5
March/April 2005

Intermediate Mathematics League of Eastern Massachusetts

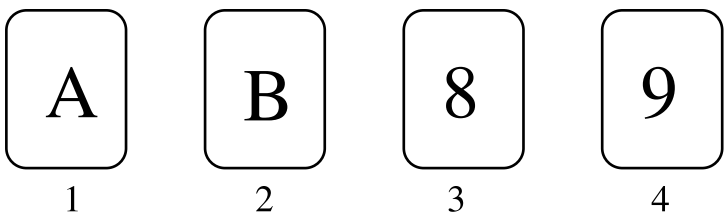
Average team score: 86.6
Average meet for the season: 95.9

Meet #5
March/April 2005

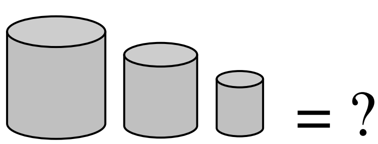
Category 1 *You may use a calculator*
Mystery
Meet #5, March/April 2005

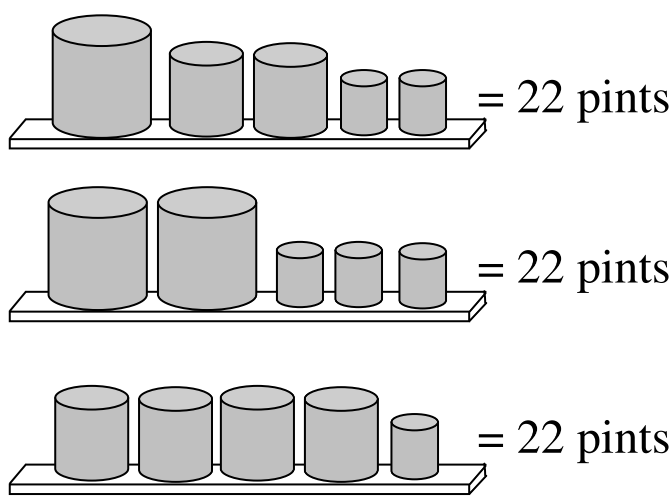
1. A triangle and a circle are drawn on the same plane. What is the greatest possible number of intersections between these two figures?

2. Each of the cards shown below has an A or a B on one side and a number on the other side. Colin has a conjecture. He believes that if there is an A on one side, then there is an even number on the other side. Which card or cards does Colin need to turn over to check his conjecture? List the numbers of the cards.



3. Cheryl's Sugar Shack sells jugs of maple syrup in three different sizes. Each jug is a whole number of pints, and jugs that appear larger hold more syrup. Cheryl has arranged the jugs of syrup on shelves so that there are 22 pints of syrup on each shelf, as shown in the picture below. If Shawn bought one of each size jug, how many pints of maple syrup did he buy?





Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 1 Average team got 17.00 points, or 1.4 questions correct
Mystery
Meet #5, March/April 2005 Average number of correct answers: 1.42 out of 3

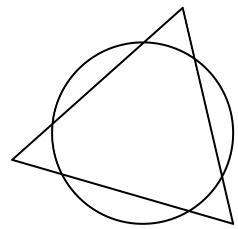
Answers

1. 6

2. 1 and 4

3. 15

1. The greatest possible number of intersections between a triangle and a circle is **6**, as shown in the diagram below.



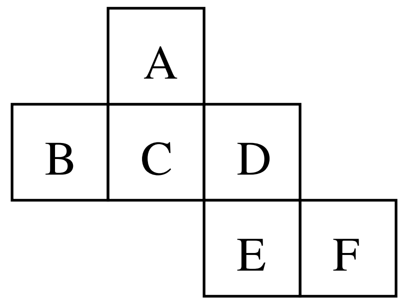
2. Card 1 must be turned over to confirm that there is an even number on the other side. Colin's conjecture makes no statement about a card with a B on it, so we don't need to turn over card 2. Nor do we care if it's an A or a B on the other side of card 3. We do need to turn over card 4, however, to make sure that an *A* does not appear on a card with an odd on the other side. Thus, Colin needs to turn over cards **1 and 4**.

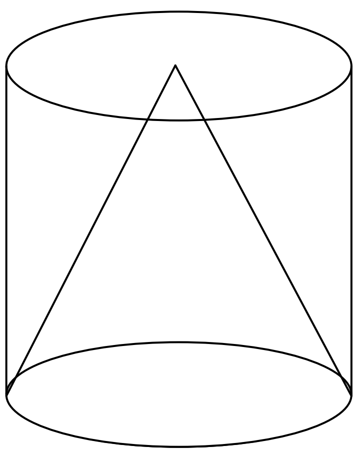
3. The shelves of maple syrup give us a system of three equations with three unknowns. Unfortunately, the middle and bottom shelves together are exactly twice the top shelf, so the system is not "linearly independent" and cannot be solved by algebra. On the other hand, if we use some other logical reasoning and some trial and error, we can solve the system. We see from the middle and bottom shelves that the smallest container must be an even number of pints. Let's try 2 pints. If the smallest container is 2 pints, then the four medium containers on the bottom shelf must account for 20 pints, so they must be 5 pints each. Similarly, the two large containers on the middle shelf must account for 16 pints, so they must be 8 pints each. Checking these values on the top shelf, we find that 8 + 5 + 5 + 2 + 2 is indeed 22. This means that Shawn must have bought 8 + 5 + 2 = **15** pints of maple syrup.

Category 2
Geometry
Meet #5, March/April 2005

You may use a calculator

1. The figure at right is a "net" for a cube. The net is folded up to form a cube, and the cube is rolled. If C is on top, what letter is on bottom?

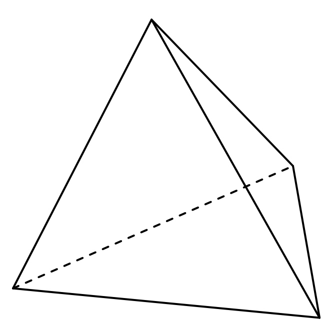




2. A solid cone with radius 2.25 cm and height 4.5 cm is placed inside a cylinder that also has radius 2.25 cm and height 4.5 cm. How many cubic centimeters of space inside the cylinder are *not* occupied by the cone? Use 3.14 for π and give your answer to the nearest tenth of a cubic centimeter.

3. How many square inches are in the surface area of a tetrahedron with side length 1.5 inches? Reminder: A tetrahedron is a solid with four congruent faces, each of which is an equilateral triangle. Hint: You can use the Pythagorean Theorem to find the height of an equilateral triangle. Give your answer to the nearest tenth of a square inch.

Answers
1. ________________
2. ________________
3. ________________



Solutions to Category 2 Average team got 18.04 points, or 1.5 questions correct
Geometry
Meet #5, March/April 2005 Average number of correct answers: 1.50 out of 3

Answers

1. F

2. 47.7

3. 3.9

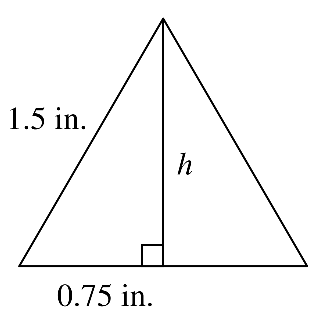
1. Letters A, B, and D are clearly adjacent to letter C, so none of these will be on the bottom of the cube. Letter E will also be adjacent to C, when the net is folded into a cube. Letter **F** will be on the opposite side of C and therefore on the bottom when C is on top.

2. The volume of a cone is one third of the volume of the cylinder, so the space *not* occupied by the cone is two thirds of the cylinder. This volume is given by the calculations below:

$$V = \frac{2}{3} \cdot \pi \cdot r^2 \cdot h = \frac{2}{3} \cdot 3.14 \cdot 2.25^2 \cdot 4.5 = 47.68875$$

To the nearest tenth of a cubic centimeter, the space inside the cylinder *not* occupied by the cone is **47.7** cubic centimeters.

3. We need to find the area of one equilateral triangle and then multiply by 4. To find the area of an equilateral triangle, we need to have a height. The picture at left shows how an equilateral triangle can be cut into two 30-60-90 triangles. The height, *h*, can be calculated using the Pythagorean Theorem as follows:



$$h = \sqrt{1.5^2 - 0.75^2} = \sqrt{2.25 - 0.5625} = \sqrt{1.6875} \approx 1.299$$

The area of one of the triangles is thus

$$A_{\text{one triangle}} = \frac{1}{2} \cdot 1.5 \cdot 1.299 \approx 0.974.$$

The surface area of the tetrahedron is four times this amount, which is $4 \cdot 0.974 = 3.896$ or **3.9** square inches to the nearest tenth of a square inch.

Category 3
Number Theory
Meet #5, March/April 2005

You may use a calculator

1. Set $A = \{a, b, c, d\}$. How many subsets are there for set A? Note: A subset of set A is a set containing all, some, or none of the elements in set A.

2. Set A is the multiples of 7.
Set B is the natural numbers that leave a remainder of 2 when divided by 5.
Set C is the multiples of 13.
Set D is the natural numbers that leave a remainder of 3 when divided by 5.
Set E is the natural numbers less than 50.
How many elements are in the set $((A \cap B) \cup (C \cap D)) \cap E$?

3. Of the 125 members of the "Divers Down" scuba club, 70 have been to the coral reefs, 80 have explored the shipwreck, and 60 have visited the underwater caves. Thirty-eight have been to both the coral reefs and the shipwreck, 42 have been to both the shipwreck and the caves, and 34 have been to the coral reefs and the caves. Only 23 divers have been to all three sites. How many members of the "Divers Down" scuba club have not been to any of these sites?

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 3 Average team got 12.73 points, or 1.1 questions correct
Number Theory
Meet #5, March/April 2005 Average number of correct answers: 1.06 out of 3

Answers

1. 16

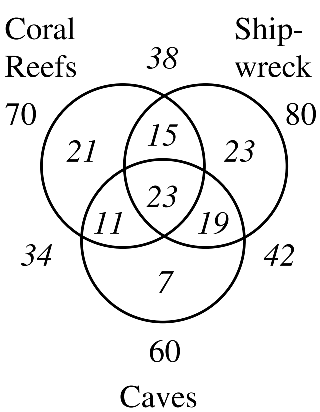
2. 3

3. 6

1. Each elements in set *A* can either be included or excluded when making a subset. This means we have two choices for each of the four letters in set *A*, so there are $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = \mathbf{16}$ possible subsets. They are: { }, {*a*}, {*b*}, {*c*}, {*d*}, {*ab*}, {*ac*}, {*ad*}, {*bc*}, {*bd*}, {*cd*}, {*abc*}, {*abd*}, {*acd*}, {*bcd*}, and {*abcd*}. The empty brackets, { }, indicate the empty set or null set. The symbol Ø is also used for the null set.

2. Set *A* is {7, 14, 21, 28, 35, 42, 49, ...}.
Set *B* is {2, 7, 12, 17, 22, 27, 32, 37, 42, 47, ...}.
Set *C* is {13, 26, 39,...}
Set *D* is {3, 8, 13, 18, 23, 28, 33, 38, 43, 48, ...}

A intersect *B* and is everything that is in both *A* and *B*, so $(A \cap B)$ is {7, 42,...}. Similarly, $(C \cap D)$ is {13,...}. The union of $(A \cap B)$ and $(C \cap D)$ is {7, 13, 42, ...}. We can end our search here, since the last step is to intersect with set *E*, the numbers less than 50. Thus, the set $((A \cap B) \cup (C \cap D)) \cap E$ has just **3** elements.



3. Since 23 divers have been to all three places and 38 have been to both the coral reefs and the shipwreck, we can determine that 38 – 23 = 15 divers have been to both the coral reefs and the shipwreck but *not* to the caves. Similarly, 19 people have been to the shipwreck and the caves but *not* to the coral reefs, and 11 have been to the coral reefs and the caves but *not* to the shipwreck. From here we can find out the numbers of divers that have been to only one of the three sites since we know the totals for each site. If we add up all seven regions of the Venn diagram shown at left, we find that only 119 divers have been to at least one site. That means there are 125 – 119 = **6** members of the club who haven't been to any of the three sites. We can get this result directly as follows
125 – 70 – 80 – 60 + 38 + 42 + 34 – 23 = **6**.

Category 4
Arithmetic
Meet #5, March/April 2005

You may use a calculator

1. A group of 8 astronauts has been training for a mission to mars, but only 5 of them will actually get to go to mars. How many different groups of 5 astronauts can be chosen from the 8 astronauts?

2. Peter has a fair number cube with whole numbers 1 through 6 and a fair spinner with whole numbers 1 through 10. If he rolls the cube and spins the spinner, what is the probability he gets a prime number in both places?
Express your answer as a reduced fraction.

3. Señor Generosa is going to give away a total of $28 to students in his math class. For each proper factor of 28, there will be a prize of that number of dollars. He will pick names at random from a hat, without replacement, and the prizes will be given out in order from least to greatest. If there are 28 students in Señor Generosa's class, how many different ways can the money be given out?
(Reminder: A proper factor of a number is any factor besides the number itself.)

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 4 Average team got 12.58 points, or 1.0 questions correct
Arithmetic
Meet #5, March/April 2005 Average number of correct answers: 1.05 out of 3

Answers

1. 56

2. $\frac{1}{5}$

3. 11,793,600

1. The situation calls for the combination 8 choose 5, which can be calculated as follows:

$$ {}_8C_5 = \frac{8!}{(8-5)! \cdot 5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \mathbf{56} $$

2. The prime numbers on the number cube are 2, 3, and 5, which is 3 out of 6 numbers. The prime numbers on the spinner are 2, 3, 5, and 7, which is 4 out of 10 numbers. The probability that he gets a prime number on both the number cube and the spinner is the product of the individual probabilities. Thus, we get

$$ \frac{3}{6} \cdot \frac{4}{10} = \frac{1}{2} \cdot \frac{2}{5} = \mathbf{\frac{1}{5}}. $$

3. First of all, we have to look at the proper factors of 28. They are 14, 7, 4, 2, and 1. It turns out that the sum of the proper factors of 28 is 28, which is why it is called a perfect number. This means that Señor Generosa will give out all $28 to five lucky students. It does make a difference if a particular student gets $14 or $1, so this is a permutation, not a combination. So the money can be given out in the following number of ways:

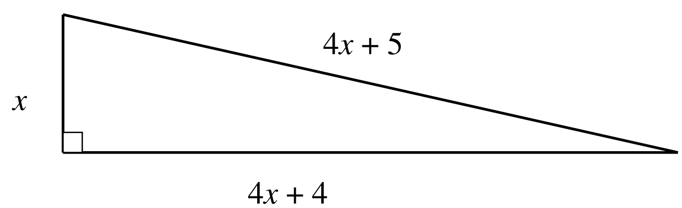
$$ {}_{28}P_5 = \frac{28!}{(28-5)!} = \frac{28!}{23!} = 28 \cdot 27 \cdot 26 \cdot 25 \cdot 24 = \mathbf{11{,}793{,}600}. $$

Category 5
Algebra
Meet #5, March/April 2005

You may use a calculator

1. A number plus six times its reciprocal is equal to five. What is the sum of the two possible numbers this could be?

2. How many units are in the length of the shorter leg in the right triangle below?



3. The graph of the equation $y = x^2 - 3x - 8$ crosses the line $y = 100$ at two different points. What is the average of the x-coordinates of these points? Express your answer as a decimal to the nearest tenth.

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 5 Average team got 11.88 points, or 1.0 questions correct
Algebra
Meet #5, March/April 2005 Average number of correct answers: 0.99 out of 3

Answers

1. 5

2. 9

3. 1.5

1. Translating the English to algebra, we get the following equation:

$$x + \frac{6}{x} = 5$$
$$x^2 + 6 = 5x$$
$$x^2 - 5x + 6 = 0$$
$$(x-2)(x-3) = 0$$

This product will be zero when $x = 2$ and when $x = 3$, so the sum of the two possible numbers is 2 + 3 = **5**.

2. Using the Pythagorean Theorem, we can write the following equation and solve for x.

$$x^2 + (4x+4)^2 = (4x+5)^2$$
$$x^2 + 16x^2 + 32x + 16 = 16x^2 + 40x + 25$$
$$x^2 + 32x + 16 = 40x + 25$$
$$x^2 - 8x - 9 = 0$$
$$(x-9)(x+1) = 0$$

The solutions are $x = 9$ or $x = -1$, but only the positive value makes sense for the side length of the triangle. The length of the shorter leg is thus **9** units.

3. Where the graph of the equation $y = x^2 - 3x - 8$ crosses $y = 100$, we will have $100 = x^2 - 3x - 8$. Solving this new equation for x, we get:

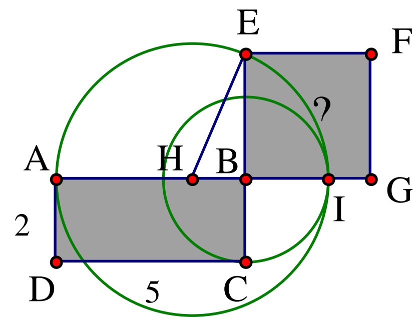
$$x^2 - 3x - 108 = 0$$
$$(x+9)(x-12) = 0$$

The two points in question are (-9, 100) and (12, 100). The average of the x-coordinates of these points is (-9 + 12)/2 = 3/2 = **1.5**.

You may use a calculator

Category 6
Team Questions
Meet #5, March/April 2005

1. In the figure at right, quadrilateral ABCD is a rectangle and quadrilateral BEFG is a square. Angle EBH is a right angle. The measure of segment DC is 5 units and the measure of segment AD is 2 units. The circle centered at B passes through points C and I, and the circle centered at H passes through points A, E, and I. How many square units are in the area of square BEFG? Express your answer to the nearest whole number.



2. Find the 20^{th} term in the sequence 0, 11, 26, 45, 68, 95, ...

3. Tildy keeps 9 pairs of gloves in a box. She has 4 identical black pairs, 3 identical brown pairs, and 2 identical blue pairs. If she picks two gloves at random, what is the probability that they are the same color and that one is a left glove and the other is a right glove? Express your answer as a reduced fraction.

4. If Rosie increases her average speed by 5 miles per hour, she will make the 780 mile trip in 1 hour less time. What is her original rate of speed? Express your answer to the nearest whole number.

5. Cylinder *A*'s radius is 2/3 the measure of cylinder *B*'s radius, and cylinder *B*'s height is 2/3 the measure of cylinder *A*'s height. What is the ratio of the volume of cylinder *A* to the volume of cylinder *B*? Express your answer as a common fraction in simplest form.

Answers

1. ______________ $= A$
2. ______________ $= B$
3. ______________ $= C$
4. ______________ $= D$
5. ______________ $= E$
6. ________________

6. Using the values the team obtained in questions 1 through 5, evaluate the following expression and round to the nearest whole number:

$$\sqrt{A + \frac{A(B+1050)E}{CD} - D}$$

Solutions to Category 6 Average team got 14.42 points, or 2.4 questions correct
Team Questions
Meet #5, March/April 2005 Average number of correct answers: 2.40 out of 6

Answers

1. 10

2. 893

3. $\frac{29}{153}$

4. 60

5. $\frac{2}{3}$

6. 33

1. The radius of the big circle is the average of the length and the width of rectangle ABCD, which is (2 + 5)/2 = 3.5 units. Leg BH of the right triangle EBH is thus 5 – 3.5 = 1.5 units long. We can determine the length of leg BE using the Pythagorean Theorem as follows:

$$m\overline{\text{BE}} = \sqrt{3.5^2 - 1.5^2} = \sqrt{12.25 - 2.25} = \sqrt{10}$$

The area of square BEFG is thus $\left(\sqrt{10}\right)^2 = \mathbf{10}$ square units. The diagram shows the construction of a square with area equal to that of a given rectangle. The length of segment BE is the square root of the product of the length and width of the rectangle, and is known as the geometric mean.

2. Rather than work our way up to the 20th term, we will find the function rule. The amount the terms in the sequence are increasing by is increasing by 4. This means we have a second degree equation. We should subtract the square of the term number from each term and see what we get. This gives the sequence –1, 7, 17, 29, 43, 59, etc.. The amount these terms are going up by is still going up by 2. Let's subtract *twice* the square of the term number from each term and see what we get. Now we get the sequence –2, 3, 8, 13, 18, 23, etc, which is no longer second degree. These terms are just increasing by 5 each time, which is linear. So far, the function rule is $f(n) = 2n^2 + 5n + C$. Let's plug in $n = 1$ and $f(1) = 0$ to solve for C. We get $f(1) = 2 \cdot 1^2 + 5 \cdot 1 + C = 0$, so C = -7. The function rule is thus $f(n) = 2n^2 + 5n - 7$. Now we can compute the 20th term directly as follows:

$$f(20) = 2 \cdot 20^2 + 5 \cdot 20 - 7 = 800 + 100 - 7 = \mathbf{893}.$$

3. We need to consider each glove color separately. Tildy has 9 pairs of gloves, or 18 gloves in all. She has an 8/18 chance of picking a black glove on the first pick. On the second pick, she has only a 4/17 chance of getting a black glove for the other hand. Thus, the probability of getting a black pair of gloves in two random picks is $\frac{8}{18}\cdot\frac{4}{17}=\frac{4}{9}\cdot\frac{4}{17}=\frac{16}{153}$. Similarly, the probability of getting a brown pair of gloves is $\frac{6}{18}\cdot\frac{3}{17}=\frac{3}{9}\cdot\frac{3}{17}=\frac{9}{153}$, and the probability of getting a blue pair of gloves is $\frac{4}{18}\cdot\frac{2}{17}=\frac{2}{9}\cdot\frac{2}{17}=\frac{4}{153}$. In all, the probability of getting a pair of gloves, of any color, in two random picks is $\frac{16}{153}+\frac{9}{153}+\frac{4}{153}=\mathbf{\frac{29}{153}}$.

4. Distance divided by rate gives time, so the time it would take Rosie at her original speed is 780/*R*. The time for the same trip at the faster rate would be 780/(*R* + 5) and is one hour less. We can write the equation 780/*R* = 780/(*R* + 5) + 1. Now we try to solve for *R*, the original rate, as shown at right. (Instead of factoring, one can use the quadratic equation.) The original rate was **60** miles per hour.

$$\begin{aligned}\frac{780}{R}&=\frac{780}{R+5}+1\\ \frac{780}{R}&=\frac{780+R+5}{R+5}\\ 780(R+5)&=R(780+R+5)\\ 780R+3900&=780R+R^2+5R\\ 3900&=R^2+5R\\ R^2+5R-3900&=0\\ (R+65)(R-60)&=0\\ R&=60 \text{ or } -65\end{aligned}$$

5. The formula for the volume of a cylinder is the area of the base times the height, or $V_{\text{cylinder}}=\pi r^2 h$. Let r_A indicate the radius of cylinder *A*, h_A the height of cylinder *A*, etc. We are looking for the ratio of the volume of cylinder *A* to the volume of cylinder *B*, or $\frac{V_{\text{A}}}{V_{\text{B}}}=\frac{\pi r_A^2 h_A}{\pi r_B^2 h_B}=\frac{r_A^2 h_A}{r_B^2 h_B}$. We know that $r_A=\frac{2}{3}\cdot r_B$ and $h_B=\frac{2}{3}\cdot h_A$. Substituting these values in the ratio, we get

$$\frac{\left(\frac{2}{3}r_B\right)^2\cdot h_A}{r_B^2\cdot\frac{2}{3}h_A}=\frac{\frac{4}{9}r_B^2\cdot h_A}{r_B^2\cdot\frac{2}{3}h_A}=\frac{\frac{4}{9}}{\frac{2}{3}}=\frac{4}{9}\cdot\frac{3}{2}=\mathbf{\frac{2}{3}}$$

6. Substituting the correct values for *A* through *E* into the expression, we get a final value of **33** as shown at right.

$$\begin{aligned}\sqrt{A+\frac{ABE}{CD}-D}&=\sqrt{10+\frac{10\cdot(893+1050)\cdot(2/3)}{(29/153)\cdot 60}-60}\\ &=\sqrt{10+\frac{12953.\overline{3}}{11.3725}-60}\\ &=\sqrt{10+1139-60}\\ &=\sqrt{1089}\\ &=\mathbf{33}\end{aligned}$$