Meet #3
January 2005

Intermediate Mathematics League of Eastern Massachusetts

Average team score: 97.4
Average meet for the season: 95.9

Meet #3
January 2005

Category 1
Mystery
Meet #3, January 2005

1. Brian is thinking of a two-digit whole number. The number is one more than a multiple of 7. The sum of the digits is 11. The tens digit is larger than the ones digit. What number is Brian thinking of?

2. How many ways can Ilian choose 3 of his 5 best friends to come with him to a movie?

3. Kaelyn has a set of four ternary weights for measuring colored sand on a balance scale. The weights are 1 gram, 3 grams, 9 grams, and 27 grams. She can weigh any whole number of grams from 1 to 40 with these four weights. For example, she can measure 5 grams of sand by placing the 9-gram weight on the left pan and the 1-gram and 3-gram weights on the right pan. She then adds the sand to the right pan until the scale is in balance. If Kaelyn measures 25 grams of sand, which of the four weights is not used at all?

Answers
1. ________________
2. ________________
3. The _____ gram weight.

Solutions to Category 1 Average team got 23.61 points, or 2.0 questions correct
Mystery
Meet #3, January 2005

Answers

1. 92

2. 10

3. 9

1. Let's start by listing two-digit numbers that are one more than the multiples of 7: 15, 22, 29, 36, 43, 50, 57, 64, 71, 78, 85, 92, and 99. Only two of these, 29 and 92, have a sum of digits equal to 11. The tens digit is greater than the ones digit for 92, so Brian must be thinking of **92**.

2. This is a combination not a permutation, since the order in which the friends are chosen does not matter. There would be $5 \times 4 \times 3 = 60$ ways to choose 3 of the 5 best friends if the order mattered. We divide this by the $3 \times 2 \times 1 = 6$ ways the same three people can be chosen. Thus there are $60 \div 6 =$ **10** different ways Ilian can choose 3 of his 5 best friends to go to a movie.

3. To measure 25 grams of colored sand, Kaelyn can put the 27-gram and the 1 gram weight on the left pan. The 3-gram weight can go on the right pan with the sand. The equation looks like this:

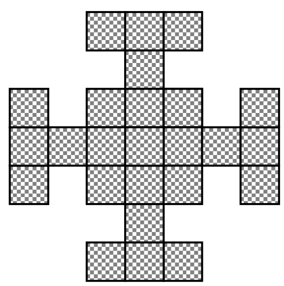
27 g weight + 1 g weight = 3 g weight + 25 g sand

In other words, the difference between the total of the weights on the left and the total of the weights on the right is made up for by the sand. The **9**-gram weight is not used at all.

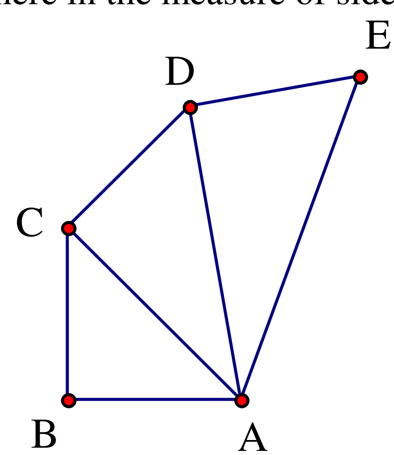
Category 2
Geometry
Meet #3, January 2005

1. A certain polygon has twice as many diagonals as sides. How many sides are there on this polygon?
Note: A diagonal in a polygon is any line segment that connects two vertices and is not a side.

2. The figure below shows the design of a raft which floats in the middle of a lake. The raft is made of a number of square sections that are linked together. If the perimeter of the raft is 220 feet, how many square feet are in the area of the raft?



3. In the figure below, triangles ABC, ACD, and ADE are right triangles, and sides AB, BC, CD, and DE have the same measure. If the measure of side AB is 2 centimeters, how many centimeters are there in the measure of side AE?



Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 2 Average team got 18.63 points, or 1.55 questions correct
Geometry
Meet #3, January 2005

Answers

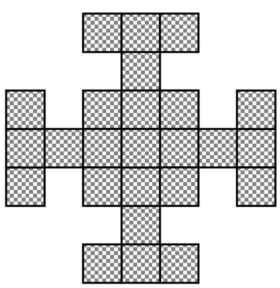
1. 7

2. 625

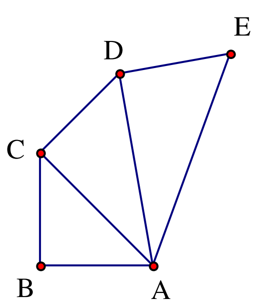
3. 4

1. The table below shows the number of diagonals for several polygons. The heptagon with **7** sides has 14 diagonals, which is twice the number of sides.

Polygon	Sides	Diagonals
Triangle	3	0
Square	4	2
Pentagon	5	5
Hexagon	6	9
Heptagon	7	14
Octagon	8	20



2. There are 44 side lengths of the square sections in the perimeter of the raft, so each square must have a side length of 220 feet ÷ 44 = 5 feet. The area of each square is thus 5 feet × 5 feet = 25 square feet.
The raft is made of 25 sections, so the area of the raft is 25 × 25 square feet = **625** square feet.



3. We will have to use the Pythagorean Theorem three times to calculate the length of each hypotenuse. Let the length of AC = x, the length of AD = y, and the length of AE = z. Then we find x as follows:

$$x^2 = 2^2 + 2^2 \Rightarrow x^2 = 8 \Rightarrow x = \sqrt{8} \Rightarrow x = 2\sqrt{2}$$

We find y as follows:

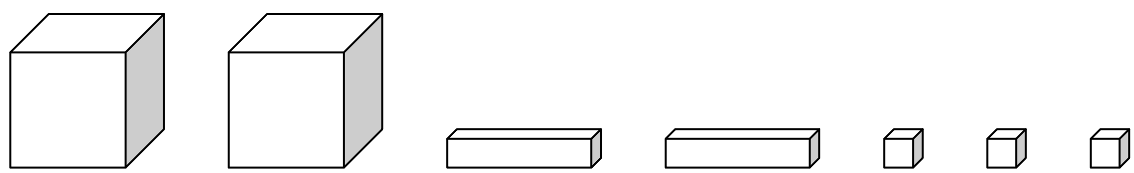
$$y^2 = 2^2 + \left(\sqrt{8}\right)^2 \Rightarrow y^2 = 12 \Rightarrow y = \sqrt{12} \Rightarrow y = 2\sqrt{3}$$

Finally, we find z as follows:

$z^2 = 2^2 + \left(\sqrt{12}\right)^2 \Rightarrow z^2 = 16 \Rightarrow z = 4$. So the length of AE is **4** centimeters.

Category 3
Number Theory
Meet #3, January 2005

1. A set of base four blocks includes unit cubes, longs, flats, and blocks. Each unit cube is 1 cm by 1 cm by 1 cm. Each "long" is 1 cm by 1 cm by 4 cm. Each "flat" is 1 cm by 4 cm by 4 cm. And each "block" is 4 cm by 4 cm by 4 cm. The base four number 2023 is given by the picture below. What is the base ten value of $2023_{\text{base four}}$?



2. Evaluate the following expression. Write your result in scientific notation.

$$\frac{1.26\times10^{9}}{4.2\times10^{-4}}$$

3. Find the base six product of the two base six numbers 541 and 23. In other words, evaluate $541_{\text{base six}}\times23_{\text{base six}}$ in base six.

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 3 Number Theory Meet #3, January 2005 .

Average team got 12.71 points, or 1.1 questions correct
The best team score was 28 out of 36 points.

Answers

1. 139

2. 3×10^{12}

3. 22123

1. The base ten value of $2023_{\text{base four}}$ is $2 \times 64 + 0 \times 16 + 2 \times 4 + 3 \times 1 = 128 + 8 + 3 =$ **139**. In other words, you would need 139 unit cubes to make two blocks that are $4 \times 4 \times 4$, two longs that are $1 \times 1 \times 4$, and three unit cubes, as shown in the picture.

2. The expression can be evaluated as follows:

$$\frac{1.26\times10^{9}}{4.2\times10^{-4}} = \frac{1.26}{4.2}\times10^{9}\times10^{4} = 0.3\times10^{13} = \mathbf{3\times10^{12}}.$$

Base Six Multiplication

$$\begin{array}{r} {}^{2} \\ 541 \\ \times\ \ 23 \\ \hline 2503 \\ +\ 15220 \\ \hline \mathbf{22123} \end{array}$$

3. We can compute the product of two base six numbers directly in base six using the standard multiplication algorithm. We just have to remember to "carry" for a group of six instead of ten. In the multiplication at left, we start with 3 times 1, which is 3. Then we multiply the 3 by the 4 in 541. Three times four is normally twelve, but in base six we get two groups of six with zero left over. We write 0 and carry 2. Next we multiply the 3 by the 5 in 541 and add the 2 we carried. That's normally seventeen, but in base six we get two groups of six with five left over. The process continues in the expected way for the digit 2 in 23. Then we add the two partial products by base six addition to get the correct product **22123**.

Base Six Multiplication

$$\begin{array}{r} 205 \\ \times\ \ 15 \\ \hline 1025 \\ +\ 2050 \\ \hline 3075 \end{array}$$

We can get the same result if we convert each number to base ten, multiply, and then convert the result back to base six, but it's much more work. This method is shown below and at left.

$$541_{\text{base six}} = 5\times36+4\times6+1\times1 = 180+24+1 = 205_{\text{base ten}}$$

$$23_{\text{base six}} = 2\times6+3\times1 = 12+3 = 15_{\text{base ten}}$$

$$3075_{\text{base ten}} = 2\times1296+2\times216+1\times36+2\times6+3\times1 = 12+3 = \mathbf{22123}_{\text{base six}}$$

Category 4
Arithmetic
Meet #3, January 2005

1. Evaluate the expression $\sqrt[4]{xy}$ if $x = 24$ and $y = 54$.

2. Evaluate the expression below. Write your result as a simplified fraction.

$$\left(\frac{2}{3}\right)^{-3} \cdot \left(\frac{9}{16}\right)^{-2} \cdot \left(\frac{7}{8}\right)^{0} \cdot \left(\frac{1}{4}\right)^{3}$$

3. How many whole numbers are there between $\sqrt[3]{25}$ and $\sqrt[3]{2005}$?

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 4 Arithmetic
Meet #3, January 2005

Average team got 12.24 points, or 1.0 questions correct
The best team score was 32 out of 36 points.

Answers

1. 6

2. $\frac{1}{6}$

3. 10

1. The expression can be evaluated as follows:

$$\sqrt[4]{xy} = \sqrt[4]{24 \cdot 54} = \sqrt[4]{(2 \cdot 2 \cdot 2 \cdot 3) \cdot (2 \cdot 3 \cdot 3 \cdot 3)} =$$

$$\sqrt[4]{2^4 \cdot 3^4} = 2 \cdot 3 = \mathbf{6}.$$

2. The expression can be evaluated as follows:

$$\left(\frac{2}{3}\right)^{-3} \cdot \left(\frac{9}{16}\right)^{-2} \cdot \left(\frac{7}{8}\right)^{0} \cdot \left(\frac{1}{4}\right)^{3} = \left(\frac{3}{2}\right)^{3} \cdot \left(\frac{16}{9}\right)^{2} \cdot 1 \cdot \left(\frac{1}{4}\right)^{3}$$

$$= \frac{3^3}{2^3} \cdot \frac{16^2}{9^2} \cdot \frac{1}{4^3} = \frac{3^3}{2^3} \cdot \frac{\left(2^4\right)^2}{\left(3^2\right)^2} \cdot \frac{1}{2^6} = \frac{3^3}{2^3} \cdot \frac{2^8}{3^4} \cdot \frac{1}{2^6} = \frac{1}{2 \cdot 3} = \mathbf{\frac{1}{6}}$$

3. The value of $\sqrt[3]{25}$ is greater than 2 but less than 3, since $2^3 = 8$ and $3^3 = 27$. Similarly, the value of $\sqrt[3]{2005}$ is greater than 12 but less than 13, since $12^3 = 1728$ and $13^3 = 2197$. Thus, the whole numbers between $\sqrt[3]{25}$ and $\sqrt[3]{2005}$ are 3 through 12, which is ten (**10**) numbers.

Category 5
Algebra
Meet #3, January 2005

1. Find the positive difference between the two solutions of the equation below.

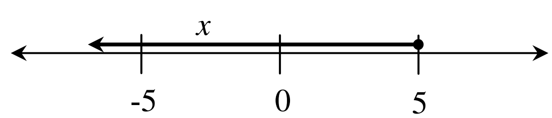
$$|6x - 4| = 36$$

2. How many integer values of *n* make the following inequality a true statement?

$$5 < \left|\frac{28}{n}\right|$$

3. For what value of *B* is the solution set of the equation below given by the graph below?

$$3(4x + 5) - 9x + B \leq 48$$



Answers

1. ________________
2. ________________
3. ________________

Solutions to Category 5 Average team got 15.88 points, or 1.3 questions correct
Algebra
Meet #3, January 2005

Answers

1. 12

2. 10

3. 18

1. To solve the equation $|6x-4|=36$, we need to solve for both the positive and the negative results.

$$6x-4=36 \qquad 6x-4=-36$$
$$6x=40 \qquad 6x=-32$$
$$x=\frac{40}{6} \qquad x=\frac{-32}{6}$$

The positive difference between these two solutions is

$$\frac{40}{6}-\left(\frac{-32}{6}\right)=\frac{40}{6}+\frac{32}{6}=\frac{72}{6}=\mathbf{12}$$

2. The inequality $5<\left|\frac{28}{n}\right|$ is true for the following ten (**10**) integer values of *n*: -5, -4, -3, -2, -1, 1, 2, 3, 4, 5.

3. The solution set given by the line graph is $x \le 5$. If we solve the inequality for *x*, we get:

$$12x+15-9x+B \le 48$$
$$3x+15+B \le 48$$
$$3x \le 33-B$$
$$x \le \frac{33-B}{3}$$
$$x \le 11-\frac{B}{3}$$

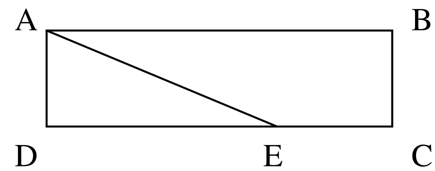
We now want to know when 11 – *B*/3 is equal to 5. We can solve the related equation 11 – 5 = *B*/3, for which *B* must be **18**.

Category 6
Team Questions
Meet #3, January 2005

1. How many regular polygons have interior angles with measures in the range 150 degrees to 160 degrees? Include the endpoints of the range in your count.

2. If 3 people can complete a 1000-piece puzzle in 8 hours, how long would it take 4 people to complete the same puzzle? Assume the extra person will work at the same average speed as the others and give your answer to the nearest whole number of hours.

3. The figure at right, the area of rectangle ABCD is 3 times the area of triangle ADE. The measure of length AD is 5 cm and the measure of length CE is 6 cm. How many centimeters are in the perimeter of trapezoid ABCE?



4. Kaelyn now has eight ternary weights for measuring colored sand on a balance scale as in Mystery problem 3. The weights are 1 gram, 3 grams, 9 grams, etc. up to 3^7 grams. She measures different amounts of sand by placing weights on the left pan and on the right pan with the sand. She then adjusts the amount of sand on the right pan until the pans are in balance. How many of the weights will Kaelyn need to put on the right pan with the sand to measure 500 grams of sand?

5. When the year 2005 is converted to a base two number, the number has eleven digits. In how many different bases is the number $2005_{\text{base ten}}$ still a four-digit number? (You should include base ten in your count.)

Answers
1. ______________ $= A$
2. ______________ $= B$
3. ______________ $= C$
4. ______________ $= D$
5. ______________ $= E$
6. __________________

6. Using the values the team obtained in questions 1 through 5, evaluate the following expression:

$$\sqrt[3]{2A^2BCDE}$$

Solutions to Category 6 Average team got 15.88 points, or 2.6 questions correct
Team Questions
Meet #3, January 2005

Answers

1. 7

2. 6

3. 42

4. 4

5. 6

6. 84

1. A regular polygon with an *interior* angle of 150 degrees has an *exterior* angle of 30 degrees. Since the sum of all the exterior angles of a polygon must be 360 degrees (why?), there must be 360 ÷ 30 = 12 sides on this regular polygon. Similarly, the regular polygon with an an *interior* angle of 160 degrees has an *exterior* angle of 20 degrees, so there must be 360 ÷ 20 = 18 sides on this regular polygon. There are 7 numbers from 12 to 18 inclusive, so there must be **7** regular polygons with interior angle measures from 150 degrees to 160 degrees.

2. Apparently, this 1000-piece puzzle takes 24 person-hours to complete (3 people × 8 hours = 24 person-hours). Thus it will take 4 people only **6** hours (4 people × 6 hours = 24 person-hours).

3. Let's call the measure of the unknown length DE x. The area of rectangle ABCD would be given by $5(x+6)$, and the area of triangle ADE would be given by $\frac{1}{2}\cdot 5x$. Since the area of rectangle ABCD is 3 times the area of triangle ADE, we can write the following equation and solve for x:

$$\begin{aligned} 5(x+6) &= 3\left(\tfrac{1}{2}\cdot 5x\right) \\ 5x+30 &= 7.5x \\ 30 &= 2.5x \\ 12 &= x \end{aligned}$$

Thus the measure of length DE is 12 cm. We need to use the Pythagorean Theorem to find the length of AE.
$h^2 = 5^2 + 12^2 \Rightarrow h^2 = 25 + 144 \Rightarrow h^2 = 169 \Rightarrow h = 13$ cm.
Now we know that AB = 18 cm, BC = 5 cm, CE = 6 cm, and DE = 13 cm, so the perimeter of trapezoid ABCE is 18 + 5 + 6 + 13 = **42** centimeters.

4. First let's find the first few powers of 3 until we get a value greater than 500: 1, 3, 9, 27, 81, 243, 729. We need to add and subtract using these numbers until we get 500. Start with 729. Too much. 729 – 243 = 486. Too little. 729 – 243 + 27 = 513. Too much. 729 – 243 + 27 – 9 = 504. Still too much. 729 – 243 + 27 – 9 – 3 = 501. Still too much. 729 – 243 + 27 – 9 – 3 – 1 = 500. Just right. So the 729-gram weight and the 27-gram weight need to go on one of the pans by themselves. On the other pan, Kaelyn needs to place the 243-gram, the 9-gram, the 3-gram, and the 1-gram weights and the 500 grams of sand. She will need to put **4** weights on the pan with the sand.

5. When the year 2005 is written in each of the six (**6**) bases shown in the table below, it remains a four-digit number:

Base	**Equivalent of 2005**
7	5563
8	3725
9	2667
10	2005
11	1563
12	11E1

6. Substituting correct values for *A* through *E* into the expression, we get:

$$\begin{aligned}\sqrt[3]{2A^2BCDE} &= \sqrt[3]{2\cdot 7^2\cdot 6\cdot 42\cdot 4\cdot 6}\\ &= \sqrt[3]{2\cdot(7\cdot 7)\cdot 6\cdot(6\cdot 7)\cdot(2\cdot 2)\cdot 6}\\ &= \sqrt[3]{(2\cdot 2\cdot 2)\cdot(6\cdot 6\cdot 6)\cdot(7\cdot 7\cdot 7)}\\ &= 2\cdot 6\cdot 7\\ &= \mathbf{84}\end{aligned}$$