Meet #5
April 2004

Intermediate Mathematics League of Eastern Massachusetts

Average team score this meet: 102
Average meet for the season: 95

Meet #5
April 2004

You may use a calculator

Category 1
Mystery
Meet #5, April 2004

1. Hector lives in the country of Metriconia, where all measurement is done in a metric-like system, including the measurement of time. The base unit of time in Metriconia is the day—the same day that we use. A dekaday is ten days, a hectoday is one hundred days, and a kiloday is one thousand days. A deciday is one tenth of a day, a centiday is one hundredth of a day, and a milliday is one thousandth of a day. If Hector just turned 45 hectodays old, how many years old is he? Round to the nearest whole number.

2. How many blocks will there be in figure 5?

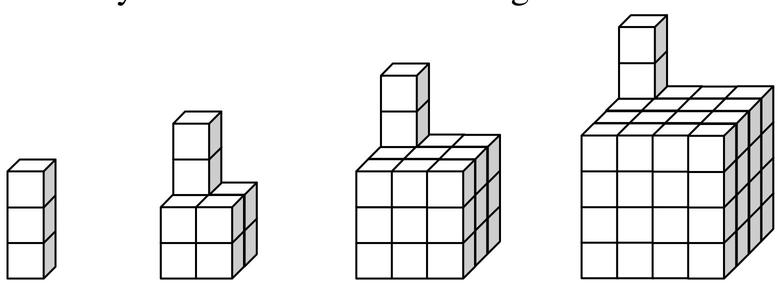


Figure 1 Figure 2 Figure 3 Figure 4

3. Kim has three thermoses that she uses to bring hot soup to school for lunch. One thermos is red, one is blue, and one is green. The thermoses have interchangeable lids, which means Kim can put any color lid on any color thermos. If Kim (who, by the way, is blind) unloads the dishwasher and randomly puts a lid on each of the three thermoses, what is the probability that *none* of the three thermoses has a matching lid? Express your answer as a simple fraction in lowest terms.

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 1 Average team got 22.2 points, or 1.85 questions correct
Mystery
Meet #5, April 2004

Answers

1. 12

2. 127

3. $\frac{1}{3}$

1. Forty-five hectodays is 4500 days. If we divide 4500 by 365 days in a year, we get 12.32876712 years. If we divide 4500 by 365.25 to include a leap day every four years, we get 12.32032854 years. Either way, we would say that he is **12** years old.

2. Each figure can be thought of as a cube with two blocks resting on top. Figure has $1^3 + 2 = 3$ blocks, figure 2 has $2^3 + 2 = 10$ blocks, figure 3 has $3^3 + 2 = 29$ blocks, figure 4 has $4^3 + 2 = 66$ blocks, and figure 5 will have $5^3 + 2 =$ **127** blocks.

3. There are six ways that three lids can be placed on three thermoses. If we use capital letters for the color of the base and small letters for the color of the lids, we can represent the six possibilities as follows:

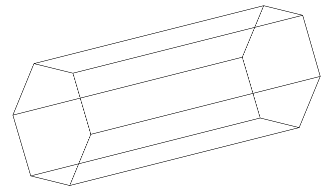
All three have matching lid:	(Rr, Bb, Gg)
One has matching lid:	(Rr, Bg, Gb); (Rb, Br, Gg); or (Rg, Bb, Gr)
None has matching lid:	(Rb, Bg, Gr); or (Rg, Br, Gb)

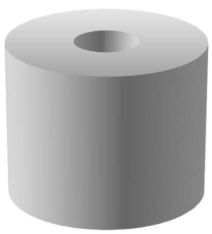
Two of the six ways have none of the lids matching, so the probability is $\frac{2}{6} = \mathbf{\frac{1}{3}}$.

Category 2
Geometry
Meet #5, April 2004

You may use a calculator

1. How many space diagonals are there in a hexagonal prism such as the one shown at right? A space diagonal is an imaginary line that connects any two vertices of a solid and passes through the interior of the solid.



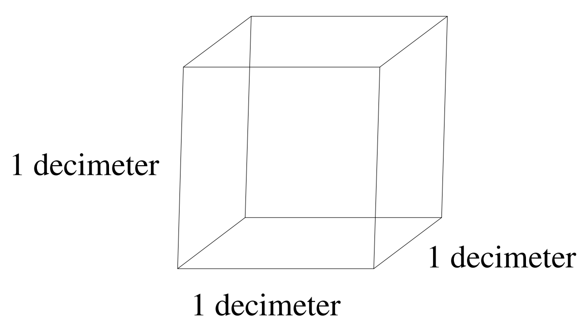


2. A roll of toilet paper has a diameter of 12 centimeters and a height of 11.4 centimeters. The inner diameter of the tube is 4 centimeters. How many cubic centimeters are in the volume of the solid part of the roll of toilet paper? Use 3.14 for Pi and round your result to the nearest tenth.

3. One cubic centimeter of water is equal to one milliliter of water. This means that a cubic decimeter, such as the one shown below, can hold one liter of water when it is filled to the brim. A steel sphere is dropped into a full cubic decimeter, displacing some water, which spills over the sides. When the ball is removed, it is noted that 732 milliliters of water remain in the cube. How many centimeters are in the diameter of the sphere? Use 3.14 for π and round your answer to the nearest whole number.

Answers

1. _______________
2. _______________
3. _______________



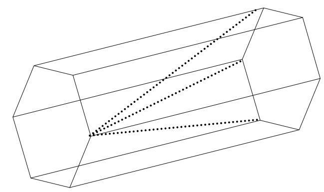
Solutions to Category 2 Average team got 9.6 points, or 0.8 questions correct
Geometry
Meet #5, April 2004

Answers

1. 18

2. 1145.5

3. 8



1. Three space diagonals can be drawn from each vertex of each base of the prism to vertices on the other base. If we imagine drawing all of these, however, we will be drawing every space diagonal twice or double counting. Therefore, we should imagine drawing all possible space diagonals from just one of the bases. There are $6 \times 3 =$ **18** space diagonals in a hexagonal prism.

2. We will compute the volume of the roll of toilet paper as if there were no hollow tube through it. Then we will subtract the volume of this hollow tube. The formula for the volume of a cylinder is $V_{\text{cylinder}} = A_{\text{base}} \times h = \pi r^2 h$. The diameter of the roll is 12 cm, so the radius is 6 cm. Using $r = 6$, $h = 11.4$ and $\pi = 3.14$ in the formula above, we get:

$$V_{\text{full cylinder}} = 3.14 \cdot 6^2 \cdot 11.4 = 1288.656$$

Similarly, we compute the volume of the hollow tube, using $r = 2$ (half of 4), $h = 11.4$ and $\pi = 3.14$:

$$V_{\text{hollow tube}} = 3.14 \cdot 2^2 \cdot 11.4 = 143.184$$

Finally, subtracting the volume of the tube from the volume of the full cylinder, we get:

$$V_{\text{roll}} = V_{\text{full cylinder}} - V_{\text{hollow tube}} = 1288.656 - 143.184 = 1145.472$$

Rounding this to the nearest tenth, we get **1145.5** cubic centimeters.

3. Since 732 millileters remain in the cube, the steel ball must have displaced $1000 - 732 = 268$ millileters. Since 1 ml^3 = 1 cm^3, we know that the volume of the sphere is 268 cubic centimeters. Substituting $V = 268$ and $\pi = 3.14$ into the formula $V_{\text{sphere}} = \frac{4}{3}\pi r^3$, we can solve for r as follows:

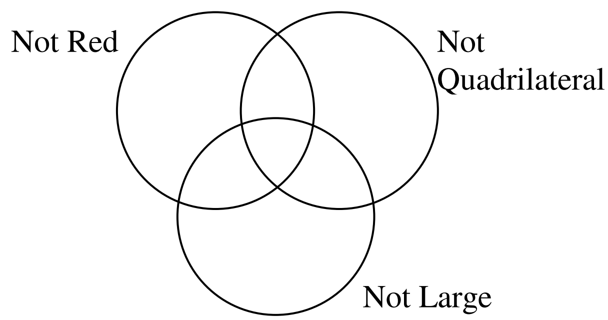
$$268 = \tfrac{4}{3} \cdot 3.14 \cdot r^3 \Rightarrow r^3 = 268 \cdot \tfrac{3}{4} \cdot \tfrac{1}{3.14} \Rightarrow r^3 = 64.01273885 \Rightarrow r \approx \sqrt[3]{64} \Rightarrow r \approx 4$$

Since $r = 4$ cm, the diameter of the steel sphere must be **8** centimeters. This may seem too large, but we should note that a cube measuring 8 cm by 8 cm by 8 cm would occupy only 8^3 or 512 cubic centimeters or a little more than half of the space.

Category 3
Number Theory
Meet #5, April 2004

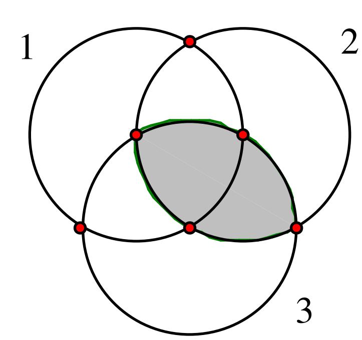
You may use a calculator

1. A set of attribute blocks contains one each of all possible combinations of four colors, five shapes, and two sizes. The colors are red, blue, green, and yellow. The shapes are triangles, squares, rectangles that aren't squares, hexagons, and circles. The sizes are large and small. If all the blocks in the set are placed according to the set rules shown in the Venn diagram below, how many blocks will be placed in the center region?



2. Set A is all the positive integer multiples of 7. Set B is all the positive integers that are 5 less than integer multiples of 8. Set C is all positive integers less than 100. How many elements are there in $(A \cup B) \cap C$?

3. Circle 1 consists of the letters in the word MATH, circle 2 consists of the letters in the word TEAM, and circle 3 consists of the letters in the word TEACHER. Find three common three-letter words that can be formed from the letters in the shaded regions of the Venn Diagram below.



Answers

1. ________________
2. ________________
3. ________________

Solutions to Category 3 Average team got 14.3 points, or 1.2 questions correct
Number Theory
Meet #5, April 2004

Answers

1. 9

2. 25

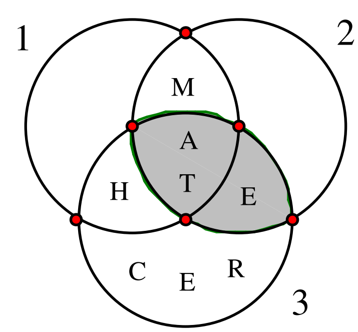
3. ATE, EAT, and TEA

1. The shapes qualifying for the center region are those that are not red, not quadrilateral, and not large. In other words, they can be any of the three other colors (blue, green, and yellow), they can be any of the three other shapes (triangles, hexagons, and circles), and they must be small. There are $3 \times 3 \times 1 =$ **9** such shapes. If you must know what they are, here is the list:

the small blue triangle,
the small blue hexagon,
the small blue circle,
the small green triangle,
the small green hexagon,
the small green circle,
the small yellow triangle,
the small yellow hexagon, and
the small yellow circle.

2. $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. In words, this means we need only consider the members of sets A and B that are less than 100. $(A \cap C) = \{7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98\}$ and $(B \cap C) = \{3, 11, 19, 27, 35, 43, 51, 59, 67, 75, 83, 91, 99\}$. There are 14 elements in $(A \cap C)$ and 13 elements in $(B \cap C)$, but they have two elements in common, namely 35 and 91. Thus, there are 14 + 13 – 2 = **25** elements in $(A \cup B) \cap C$.

3. The Venn Diagram at right shows where the letters should be placed. The three letters in the two shaded regions are A, T, and E. The three common three-letter words are **ATE**, **EAT**, and **TEA**.

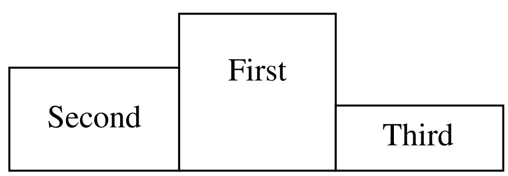


Category 4
Arithmetic
Meet #5, April 2004

You may use a calculator

1. Hadley has pizza for lunch three out of every five days and drinks chocolate milk one out of every three days. What is the probability that she has pizza and chocolate milk on any given day? Express your answer as a simple fraction in lowest terms.

2. Aaron, Beth, Cecille, Darius, Edward, Fred, Gabrielle, and Herbert plan to run in a race. At the awards ceremony, the first, second, and third place runners will stand on a platform such as the one shown here for a photograph. The photographer won't be able to attend the awards ceremony, so he had the eight runners pose for all the possible ways that three of them could place first, second, and third. How many different photographs did the photographer take if each possibility was photographed just once?



3. Aaron, Beth, Cecille, Darius, Edward, Fred, Gabrielle, and Herbert were climbing in the Himalayas. They agreed that three of the eight should stay at base camp while the other five attempt the summit. How many different groups of three people from the eight people can be chosen to stay at base camp?

Answers
1. ________________
2. ________________
3. ________________

Solutions to Category 4 Average team got 14.5 points, or 1.2 questions correct
Arithmetic
Meet #5, April 2004

Answers

1. $\frac{1}{5}$

2. 336

3. 56

1. The event that Hadley has pizza *and* drinks chocolate milk is often refered to as a compound event. We can multiply the probabilities of the individual events to find the probability of the compound event. Doing so, we get $\frac{3}{5} \times \frac{1}{3} = \mathbf{\frac{1}{5}}$.

2. There are 8 possible first place winners. For each of these eight possible first place winners, we must consider 7 possible second place winners and 6 possible third place winners. Thus, the photographer would need to take $8 \times 7 \times 6 =$ **336** different pictures. This is known as a *permutation* and also calculate as follows:

$$_8P_3 = \frac{8!}{(8-3)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 8 \cdot 7 \cdot 6 = \mathbf{336}$$

3. In choosing this group of three from the group of eight, there is no special significance to being chosen first, second, or third. In other words, the selection of Aaron, then Beth, then Cecille is no different than the selection of Aaron, then Cecille, then Beth, since the same three people will stay at base camp. If we divide the 336 from question 2 by the number of different ways that three people can be arranged, which is $3 \times 2 \times 1 = 6$, we get $336 \div 6 =$ **56** different groups of three people from the eight people. This is a *combination* problem. In the language of combinatorics, we say “eight choose three is fifty-six.” We can calculate as follows:

$$_8C_3 = \frac{8!}{(8-3)! \cdot 3!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = \mathbf{56}$$

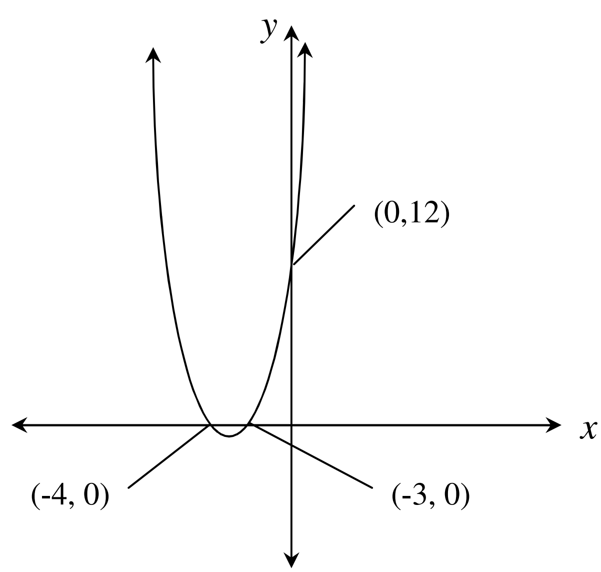
Category 5
Algebra
Meet #5, April 2004

You may use a calculator

1. The length of a rectangle is nine units longer than the width of the rectangle and the area of the rectangle is 112 square units. Find the number of units in the perimeter of the rectangle.

2. The product of three consecutive negative integers is equal to 288 times the middle integer. What is the value of the middle integer?

3. The graph of the equation $y = (x + 3)(x - a)$ is shown below. The ordered pairs of the x- and y-intercepts have also been given. Find the value of a.



Answers
1. ______________
2. ______________
3. ______________

Solutions to Category 5 Average team got 17.3 points, or 1.4 questions correct
Algebra
Meet #5, April 2004

Answers

1. 46

2. -17

3. -4

1. Let the width of the rectangle be x units. The length is then $x + 9$ units, and the area would be calculated as $x(x+9)$ or $x^2 + 9x$ square units. Since we know the area is 112 square units, we can write the quadratic equation $x^2 + 9x = 112$. Setting this equal to zero, we get $x^2 + 9x - 112 = 0$. Now we would like to factor this trinomial into the product of two binomials in the form $(x+ \quad)(x- \quad)$. To fill these blanks, we must find a pair of numbers whose product is 112 and whose difference is 9. We could list factors of 112 in pairs as follows: 1×112, 2×56, 4×28, 7×16, and 8×14. The desired pair is 7 and 16 since $16 - 7 = 9$. We now have $x^2 + 9x - 112 = (x+16)(x-7) = 0$. For this equation to equal zero, x has to equal either -16 or 7. The negative value doesn't make sense for the width of our rectangle, so the width must be 7. This means the length is $7 + 9 = 16$, and the perimeter is $2(7+16) = 2 \cdot 23 = \mathbf{46}$ units.

2. If we let n be the middle number, the product of the three consecutive numbers is $(n-1) \cdot n \cdot (n+1)$. This product is said to be equal to 288 times the middle number, so we can write the equation $(n-1) \cdot n \cdot (n+1) = 288n$. Dividing both sides of the equation by n, we avoid a cubic and get the quadratic equation $(n-1)(n+1) = 288$. Multiplying out the left side of the equation, we get $n^2 - 1 = 288$, which means that n^2 must be 289. Both positive and negative 17 satisfy the equation $n^2 = 289$, but we know that the three consecutive integers are negative. The value of the middle integer is **-17**.

3. Expanding the right side of the equation, we get $y = x^2 + (3-a)x - 3a$. If we substitute the x- and y-values from the ordered pair (0, 12) into this equation, we get $12 = 0^2 + (3-a) \cdot 0 - 3a$, which simplifies to $12 = -3a$. Dividing both sides by -3, we get $a =$ **-4**.

Category 6
Team Questions
Meet #5, April 2004

You may use a calculator

1. Nancy bought a piece of purple ribbon that measured exactly 143 centimeters in length. With incredible precision, she cut as many 1.4 centimeter pieces as she could from the ribbon and was left with a small piece of ribbon that we will call the remainder. To the nearest tenth, how many centimeters are in the length of this remainder?

2. Hector, like many students in Metriconia, spends about 7.5 centidays on his homework every evening. To the nearest whole number, how many of our minutes does Hector spend on his homework?

3. The Double Bridge Club meets every Thursday evening at the community center, where there are plenty of square card tables. Each game of bridge requires four members, and each member of the Double Bridge Club plays exactly two simultaneous games of bridge, going back and forth between the two games. Any pair of bridge games has exactly one member in common. What is the minimum number of club members who must be present on a Thursday evening for these requirements to be met?

4. Find the twentieth term of the sequence 13, 18, 25, 34, 45, 58, 73, ...

5. One parsec is about 3.26 light-years. A light-year is a measure of distance; the distance that light can travel through space in one year. Light travels at a speed of about 186,000 miles per second. Calculate the number of miles in a parsec. Express this value in scientific notation, rounded to the nearest single-digit whole number times a power of ten.

6. Using the values the team obtained in questions 1 through 5, evaluate the following expression:

$$\frac{BCD}{\sqrt[4]{\dfrac{E}{AC^2}}}$$

Answers

1. ______________ = A
2. ______________ = B
3. ______________ = C
4. ______________ = D
5. ______________ = E
6. ______________

Solutions to Category 6 Average team got 24.3 points, or 4.1 questions correct
Team Questions
Meet #5, April 2004

Answers

1. 0.2 or .2

2. 108

3. 10

4. 450

5. 2×10^{13}

6. 486

1. Dividing the full length 143 centimeters by 1.4 on most calculator gives the quotient 102.1428571, which has been rounded to fit the screen. This means that Nancy was able to cut 102 pieces of ribbon of length 1.4 cm from the original ribbon. We must realize that the decimal part of our quotient, the 0.1428571, represents the portion of another 1.4 centimeter piece of ribbon. It does *not* represent the length of this remainder. There are two ways to find the length we need. One way is to multiply this 0.1428571 by 1.4 cm, which gives 0.2 centimeters. Another way is to find out how much of the ribbon is used by the 102 pieces of length 1.4 cm, which is $102 \times 1.4 = 142.8$ cm. Then we subtract this from the total to get $143 - 142.8 =$ **0.2** cm.

2. There are $24 \times 60 = 1440$ minutes in a day. One centiday is equal to 0.01 days, so 7.5 centidays is 0.075 days. Multiplying 0.075 days by 1440 minutes in a day, we get **108** minutes.

3. We can use letters to represent the different players. Each letter must appear exactly twice and each pair of games must have exactly one letter in common. The table below shows how ten (**10**) people can play five games of double bridge. Twenty positions are played by just ten people since everyone plays two games.

Game 1	Game 2	Game 3	Game 4	Game 5
A	A	B	C	D
B	E	E	F	G
C	F	H	H	I
D	G	I	J	J

4. The numbers in the sequence are increasing and the amount by which they are increasing is increasing by 2. The table below shows the term number, the value of the terms, the first difference (D1) and the second difference (D2). The twentieth term is **450**. The function rule would be $V = T^2 + 2T + 10$, where T is the term number and V is the value of that term.

Term	**Value**	**D1**	**D2**
0	10		
1	13	3	
2	18	5	2
3	25	7	2
4	34	9	2
5	45	11	2
6	58	13	2
7	73	15	2
8	90	17	2
9	109	19	2
10	130	21	2
11	153	23	2
12	178	25	2
13	205	27	2
14	234	29	2
15	265	31	2
16	298	33	2
17	333	35	2
18	370	37	2
19	409	39	2
20	450	41	2

5. If we keep track of our units, it will be easier to check that we have made all the necessary multiplications.

$$\frac{186{,}000\text{ miles}}{\text{second}} \times \frac{60\text{ seconds}}{\text{minute}} \times \frac{60\text{ minutes}}{\text{hour}}$$

$$\times \frac{24\text{ hours}}{\text{day}} \times \frac{365.25\text{ days}}{\text{year}} \times \frac{3.26\text{ lightyears}}{\text{parsec}}$$

$$= 1.913526634 \times 10^{13}\text{ miles}$$

Rounding this to the nearest the single digit whole number times a power of ten, we get $\mathbf{2 \times 10^{13}}$ miles. (Note: 365 days per year gives the same result.)

6. Substituting the correct values A through E into the expression, we get:

$$\frac{BCD}{\sqrt[4]{\dfrac{E}{AC^2}}} = \frac{108 \cdot 10 \cdot 450}{\sqrt[4]{\dfrac{2 \cdot 10^{13}}{0.2 \cdot 10^2}}} = \frac{486{,}000}{\sqrt[4]{10^{12}}} = \frac{486{,}000}{1000} = \mathbf{486}$$