

Meet #4
February, 2003

Intermediate
Mathematics League
of
Eastern Massachusetts

www.imlem.org

Meet #4
February, 2003

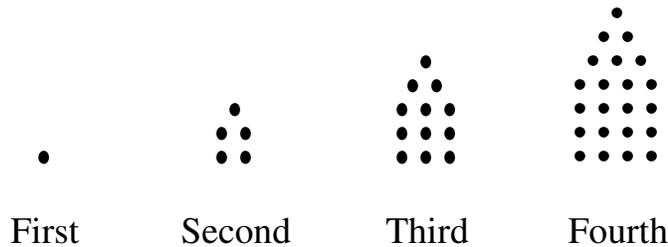
Category 1

You may use a calculator

Mystery

Meet #4, February, 2003

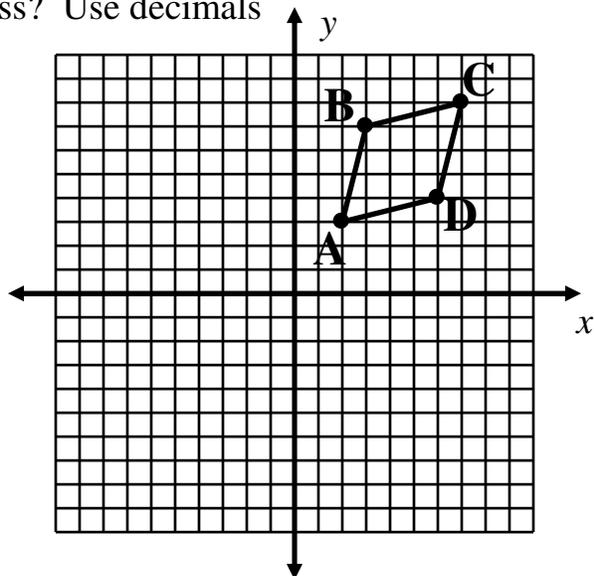
1. The numbers 1, 5, 12, and 22 are called *pentagonal numbers* because these numbers of dots (or other objects) can be arranged into pentagons as shown below. How many dots would there be in the tenth pentagonal number?



2. Solve for n :

$$\begin{array}{r} 35 \\ n \overline{)609} \end{array} \text{ Remainder } 14$$

3. Quadrilateral ABCD is reflected over the y -axis, creating the quadrilateral $A'B'C'D'$. This quadrilateral is then rotated 180 degrees about the origin to create quadrilateral $A''B''C''D''$. What are the coordinates of the point at which the diagonals of quadrilateral $A''B''C''D''$ cross? Use decimals to the nearest tenth if necessary.



Answers	
1.	_____
2.	_____
3.	(,)

Solutions to Category 1

Mystery

Meet #4, February, 2003

Answers

1. 145

2. 17

3. (4.5, -5.5)

1. One way to find the tenth pentagonal number is to continue the pattern. The amount that the numbers increase by is itself increasing by 3. The table below summarizes:

Term	1	2	3	4	5	6	7	8	9	10
Amount to Add	1	4	7	10	13	16	19	22	25	28
Number of Dots	1	5	12	22	35	51	70	92	117	145

Another way to find the tenth pentagonal number is to notice that each pattern of dots can be thought of as a square arrangement of dots with a triangular arrangement of dots on top. The square part is the square of the term number and the triangle part is the previous triangular number. In mathematical notation, we can say that $P_n = S_n + T_{n-1}$. For the tenth pentagonal number, we simply add the tenth square number (100) and the ninth triangular number (45) to get **145**.

2. Subtracting 14 from 609, we get 595, which is the product of 35 and n . Now we can just divide 595 by 35 to get the value of n , which is **17**.

$$\begin{array}{r} 35 \\ n \overline{)609} \end{array} \text{ Remainder } 14$$

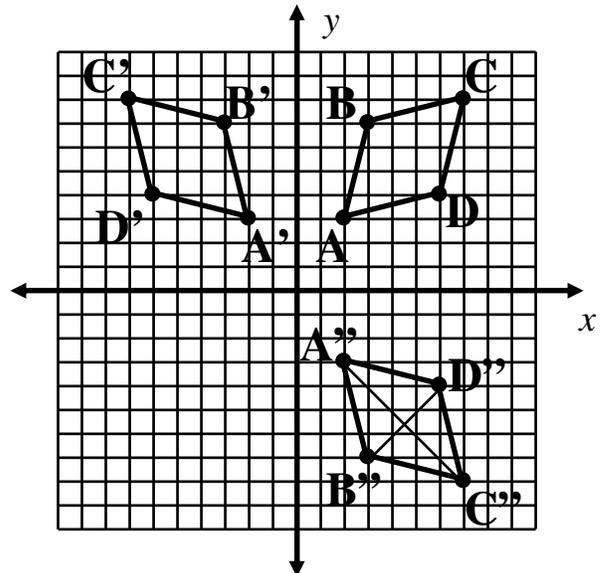
$$\Rightarrow 35n + 14 = 609$$

$$\Rightarrow 35n = 595$$

$$\Rightarrow n = 17$$

3. The coordinate system at right shows where the quadrilateral $A''B''C''D''$ will end up. The diagonals $\overline{A''C''}$ and $\overline{B''D''}$ cross at coordinates **(4.5, -5.5)**.

Thanks to Robin Du for pointing out that the answer in the answer box above was incorrect. The solution was right. 8/23/07



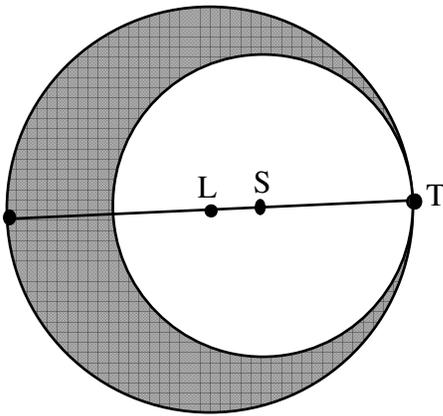
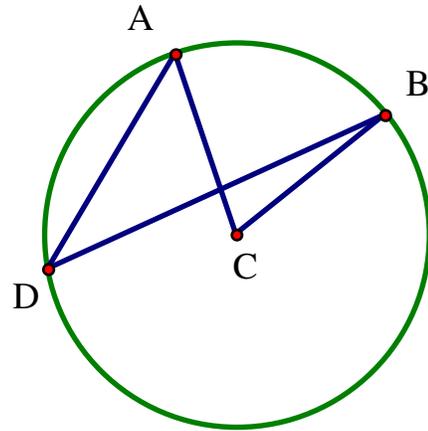
Category 2

Geometry

Meet #4, February, 2003

You may use a calculator

1. In the figure at right, the measure of minor arc AB is 70 degrees. How many degrees are in the measure of the inscribed angle ADB?



2. The two circles in the figure at left are tangent at point T, meaning that they touch at that single point. The radius of the smaller circle (ST) is three fourths the radius of the larger circle (LT) and the radius of the larger circle is one inch. How many square inches are in the area of the shaded region? Use 3.14 for π and round your answer to the nearest hundredth of a square inch.

3. Bobby rigged up a funny bicycle with a big wheel in the back and a small wheel in the front. If the back wheel has a diameter of 27 inches and the front wheel has a diameter of 24 inches, about how many more times will the front wheel go around than the back wheel during a one-mile ride? There are 5280 feet in a mile and 12 inches in a foot. Use 3.14 for π and round your answer to the nearest whole number of rotations.

Answers	
1.	_____
2.	_____
3.	_____

Solutions to Category 2
Geometry
Meet #4, February, 2003

Answers

1. 35

1. The measure of an inscribed angle is half the measure of the arc it “subtends”. Thus angle ADB measures half of arc AB which is $70 \div 2$ or **35** degrees.

2. 1.37

2. To find the area of the shaded region, we can subtract the area of the smaller circle from the area of the larger circle. The area of the larger circle is π square inches, since the radius is 1 inch and $\pi \times 1^2 = \pi$. The area of the smaller circle is $\pi \times (0.75)^2 = \pi \times 0.5625$. The difference is $\pi - 0.5625\pi = (1 - 0.5625)\pi = 0.4375\pi$. Using 3.14 and rounding to the nearest hundredth of a square inch, we get $0.4375 \times 3.14 \approx$ **1.37** square inches.

3. 93

3. The back wheel with a diameter of 27 inches will travel $27 \times 3.14 = 84.78$ inches with each revolution, or about $84.78 \div 12 = 7.065$ feet. The back wheel will rotate about $5280 \div 7.065 = 747.346$ times in one mile. The front wheel, on the other hand, with a diameter of 24 inches will travel $24 \times 3.14 = 75.36$ inches with each revolution, or about 6.28 feet, and will rotate about $5280 \div 6.28 = 840.764$ times in one mile. Thus the front wheel will rotate about $840.764 - 747.346 = 93.418$ times more than the back wheel, which is **93**, to the nearest whole number.

Category 3

You may use a calculator

Number Theory

Meet #4, February, 2003

1. Amanda bought a new remote control device for her television. It allows her to advance the channel number by fives. For example, if she's watching channel 12 and hits the "advance five" button, the television switches to channel 17. Amanda has exactly 43 channels on her television, numbered 1 to 43, and the next channel after 43 would be channel 1. If she starts out on channel 12 and presses the "advance five" button 27 times in a row, what channel will she end up watching?

2. Find the 48th term of the following sequence:

23, 32, 41, 50, ...

3. In modular arithmetic, a "unit" is a number that has one (1) as a multiple in a particular modular system. Let's say a "unit pair" is a pair of numbers whose product is one and a "unit partner" is the other number in a unit pair if one of the numbers is known. For example, in Mod 7 the product 2×4 is equal to 1, so 2 and 4 make a unit pair in Mod 7. Also, 2 is the unit partner for 4 and 4 is the unit partner for 2 in Mod 7. (A number can be its own unit partner.) What is the unit partner for 8 in Mod 19?

Answers

1. _____

2. _____

3. _____

Solutions to Category 3
Number Theory
Meet #4, February, 2003

Answers

1. 18

2. 446

3. 12

1. If Amanda hits the “advance five” button 27 times, she will advance $5 \times 27 = 135$ channels. Since she started at 12, she would be on $135 + 12 = 147$ if there were such a channel. Since the cycle starts over again after channel 43, we need to divide 147 by 43 and find the remainder: $147 = 3 \times 43 + 18$. Amanda will go through the cycle three times and land on channel **18**.

2. The terms are increasing by 9 each time. From the first term to the 48th, there will be exactly 47 increases of 9. That’s a total increase of $47 \times 9 = 423$. Adding this total increase to our first term of 23, we find that the 48th term of the sequence is **446**.

3. We are looking for a multiple of 8 that is also one more than a multiple of 19. Since we are more likely to recognize a multiple of 8, it may help to list numbers that are one more than a multiple of 19. Starting with one and adding 19’s, we get: 1, 20, 39, 58, 77, 96, etc. 96 is 12 times 8, so the unit partner of 8 is **12**.

Category 4

You may use a calculator

Arithmetic

Meet #4, February, 2003

1. How much greater is $\frac{1}{4}\%$ of 2000 than 175% of 2?
Express your answer as a mixed number in lowest terms.

2. Ralph has \$28.56 with him at the candy store and he wants to buy as much chocolate as he can. The chocolate is \$4.35 per pound and he must pay a 5% sales tax. What is the greatest number of pounds he can purchase? Round your answer to the nearest hundredth.

3. Stephanie deposited \$10,000 at a bank in an 18-month certificate of deposit account that pays an annual interest rate of 3% . How much money can Stephanie expect to have in her bank account at the end of the 18 months if the interest is compounded monthly? Express your answer to the nearest whole number of dollars.

Answers	
1.	_____
2.	_____
3.	_____

Solutions to Category 4
 Arithmetic
 Meet #4, February, 2003

Answers

1. $1\frac{1}{2}$

2. 6.25

3. \$10,460

Month	Balance
0	\$10000.00
1	\$10025.00
2	\$10050.06
3	\$10075.19
4	\$10100.38
5	\$10125.63
6	\$10150.94
7	\$10176.32
8	\$10201.76
9	\$10227.26
10	\$10252.83
11	\$10278.46
12	\$10304.16
13	\$10329.92
14	\$10355.74
15	\$10381.63
16	\$10407.59
17	\$10433.61
18	\$10459.69

1. One percent of 2000 is 20 and a quarter of 20 is 5. 175% of 2 is all of 2 plus three quarters of 2, which is 3.5. $5 - 3.5$ is 1.5. Expressing this as a mixed number,

we get $1\frac{1}{2}$.

2. Ralph will have to pay 100% of the price of the candy plus 5% for sales tax, which is 105% altogether. If we divide the amount of money he has by 1.05, we find that he can buy $\$28.56 \div 1.05 = \27.20 . Dividing this amount by \$4.35 per pound, we find that he can buy $27.20 \div 4.35 = 6.25$ pounds of chocolate.

3. The monthly interest rate will be one twelfth of the annual rate, which is $3\% \div 12 = 0.25\%$. After one month, the bank will deposit 0.25% of \$10,000, which is $0.0025 \times \$10,000 = \25 , into Stephanie's account. Her new balance will be \$10,025 and she will then earn interest on the interest as well as interest on her original deposit (her principle). A short-cut way to calculate the balance each month is to multiply the previous balance by 1.0025. The table at left summarizes. Since we will multiply by 1.0025 eighteen times in all, we can also calculate the final balance as follows:

$$\$10,000 \times 1.0025^{18} \approx \$10,000 \times 1.045969 = \$10,459.69.$$

To the nearest whole number of dollars, Stephanie will have **\$10,460** dollars in her bank account.

Solutions to Category 5

Algebra

Meet #4, February, 2003

Answers

1. 7569

2. 18

3. $5\frac{1}{16}$

1. Dividing the sum by 2 will give the middle integer. $174 \div 2 = 87$. (The first odd integer is 85 and the last is 89.) The square of 87 is $87 \cdot 87 = \mathbf{7569}$.

2. It is easy to get lost in the language of a question like this one. To solve this with the help of algebra, let's let A be Alice's age now and set up a table to keep track of the different ages at the different times.

	Last Year	Now	Five years from now
Susan	$A + 6 - 1$	$A + 6$	$A + 6 + 5$
Alice	$A - 1$	A	$A + 5$

Now we need to write an equation based on the information given. We get: $A + 6 + 5 = 2(A - 1)$. Solving this equation for A , we get:

$$A + 6 + 5 = 2(A - 1)$$

$$A + 11 = 2A - 2$$

$$A + 13 = 2A$$

$$13 = A$$

If Alice is 13 years old now, she will be **18** in five years.

3. On the way out, Heather and Scott were traveling 3 times as fast as they were on the way back. This means that the trip back took 3 times as long. If we add the one unit of time for the trip out and the 3 units of time for the trip back, we get 4 equal units of time. If we divide the total time of $2\frac{1}{4}$ hours into 4 equal parts, we get $2\frac{1}{4} \div 4 = \frac{9}{4} \div 4 = \frac{9}{4} \times \frac{1}{4} = \frac{9}{16}$ hours. If they paddled for $\frac{9}{16}$ of an hour at 9 miles per hour on the way out, then they traveled $\frac{9}{16} \times 9 = \frac{81}{16} = 5\frac{1}{16}$ miles on the way out. The island is thus $5\frac{1}{16}$ miles away.

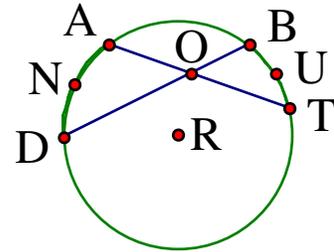
Category 6

You may use a calculator

Team Questions

Meet #4, February, 2003

1. In the circle shown here, the measure of angle BOT equals one-half the sum of arc angles AND and BUT. If the measure of arc angle AND is 53 degrees and the measure of arc angle BUT is 39 degrees, how many degrees are in the measure of angle DOT?

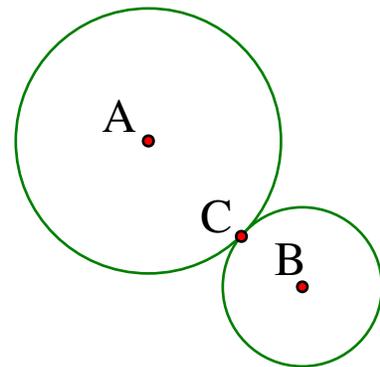


2. Compute the sum of the prime factors of $2^{16} - 1$.

3. The tenth term of an arithmetic sequence is 222 and the seventeenth term is 313. What is the value of the twenty-ninth term?

4. What is the remainder when 5^{587} is divided by 16?

5. In the figure at right, circles A and B are tangent at point C. Circle A has a radius of 1.75 cm and circle B has a radius of 1.05 cm. What is the ratio of the area of circle A to the area of circle B? Express your answer as a mixed number in lowest terms.



Answers	
1.	_____ = A
2.	_____ = B
3.	_____ = C
4.	_____ = D
5.	_____ = E
6.	_____

6. Using the values you obtained in questions 1 through 5, evaluate the following expression. Express your answer as a mixed number in lowest terms.

$$\sqrt{\frac{7C + 6B}{E} - 2(A + 1) + D}$$

Solutions to Category 6
 Team Questions
 Meet #4, February, 2003

Answers

1. 134

2. 282

3. 469

4. 13

5. $2\frac{7}{9}$

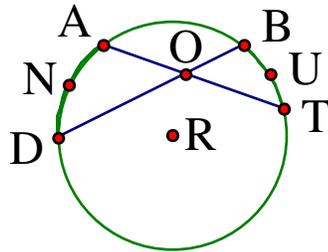
6. 52

1. First we should find the measure of angle BOT by the method described. We get:

$$m\angle BOT = \frac{m\widehat{AND} + m\widehat{BUT}}{2} = \frac{53^\circ + 39^\circ}{2} = \frac{92^\circ}{2} = 46^\circ$$

Angles BOT and DOT are supplementary, so we can find the measure of angle DOT as follows:

$$m\angle DOT = 180^\circ - m\angle BOT = 180^\circ - 46^\circ = \mathbf{134^\circ}$$



$$\begin{array}{r} 5 \overline{)65535} \\ 3 \overline{)13107} \\ 17 \overline{)4369} \\ 257 \overline{)257} \\ \quad \quad \quad 1 \end{array}$$

2. The value of $2^{16} - 1$ is 65535. Using the ladder method shown at left, we find that the prime factorization of 65535 is $3 \times 5 \times 17 \times 257$. (Note that it not necessary to test any primes greater than 17, since $17^2 = 289$ is more than 257.) The sum of these four primes is $3 + 5 + 17 + 257 = \mathbf{282}$.

3. From the tenth to the seventeenth term in an arithmetic sequence, a constant amount is added seven times. Thus, the difference $313 - 222 = 91$ accounts for those seven equal additions. Dividing 91 by 7 we get 13. To go from the seventeenth term to the twenty-ninth term, we must add 13 twelve more times to 313. $313 + 12 \times 13 = 313 + 156 = \mathbf{469}$.

	Power of
1	5
2	25
3	125
4	625
5	3125
6	15625
7	78125
8	390625
9	1953125
10	9765625
11	48828125
12	

4. The value 5^{587} is too large for most calculators to compute without rounding. One strategy is to look for patterns in the powers of five. The units digit of all powers of five is clearly a 5. From the second power on, the tens digit is always a 2. From the third power on, the hundreds digit alternates between 1 and 6—it's 1 when the power is odd and 6 when the power is even. The thousands place eventually falls into a 4-cycle pattern that goes 3, 5, 8, 0, with the zero occurring when the exponent is a multiple of four. It so happens that 10,000 is divisible by 16 (16 times 625 is 10,000). This means that we only need to know the last four digits of the value of 5^{587} to determine the remainder when this very large number is divided by 16. Since 587 is three more than a multiple of four, the pattern tells us that the last four digits of 5^{587} will be 8125. Dividing 8125 by 16, we get 507 and a decimal. Since 507 times 16 is 8112, the remainder must be $8125 - 8112 = 13$.

5. Since we will have to express our answer as a mixed number, we may as well work in fractions the whole time.

$$\text{The area of circle A is } \pi \times \left(1\frac{3}{4}\right)^2 = \pi \times \left(\frac{7}{4}\right)^2 = \frac{49}{16} \pi$$

$$\text{The area of circle B is } \pi \times \left(1\frac{1}{20}\right)^2 = \pi \times \left(\frac{21}{20}\right)^2 = \frac{441}{400} \pi$$

$$\frac{\frac{49}{16} \pi}{\frac{441}{400} \pi} = \frac{\frac{49}{16}}{\frac{441}{400}} = \frac{49}{16} \times \frac{400}{441} = \frac{25}{9} = 2\frac{7}{9}$$

$$\text{The ratio of A to B is } \frac{49}{400} \pi \quad \frac{441}{400}$$

6. Evaluating the expression with the correct values of A through E, we get:

$$\sqrt{\frac{7C+6B}{E} - 2(A+1) + D} = \sqrt{\frac{7 \times 469 + 6 \times 282}{2\frac{7}{9}} - 2(134+1) + 13}$$

$$= \sqrt{\frac{3283+1692}{\frac{25}{9}} - 2(135) + 13} = \sqrt{\frac{4975}{1} \times \frac{9}{25} - 270 + 13}$$

$$= \sqrt{199 \times 9 - 270} + 13 = \sqrt{1791 - 270} + 13 = \sqrt{1521} + 13 = 39 + 13 = 52$$