Intermediate Mathematics League of Eastern Massachusetts

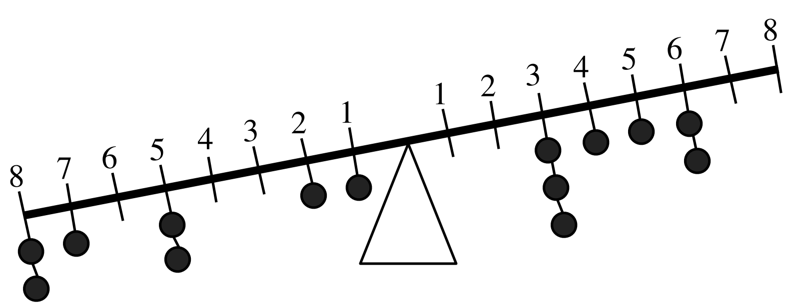
Meet #3
January, 2003

Category 1
Mystery
Meet #3, January, 2003

1. Jill is walking to school on June 2nd and it occurs to her that there will be five Sundays this June. If the last day of school is June 24th, what day of the week will the last day of school be?

2. Dale paid $520 for new skis, bindings, and boots. The boots cost twice as much as the skis and the skis cost three times as much as the bindings. How many dollars did Dale pay for the skis?

3. In order for the beam below to balance, the total moment on the left must equal the total moment on the right. (A *moment* is the weight of an object times its distance from the balance point.) If all the balls weigh the same, under which number on the right could you place a single ball to bring the beam into balance?



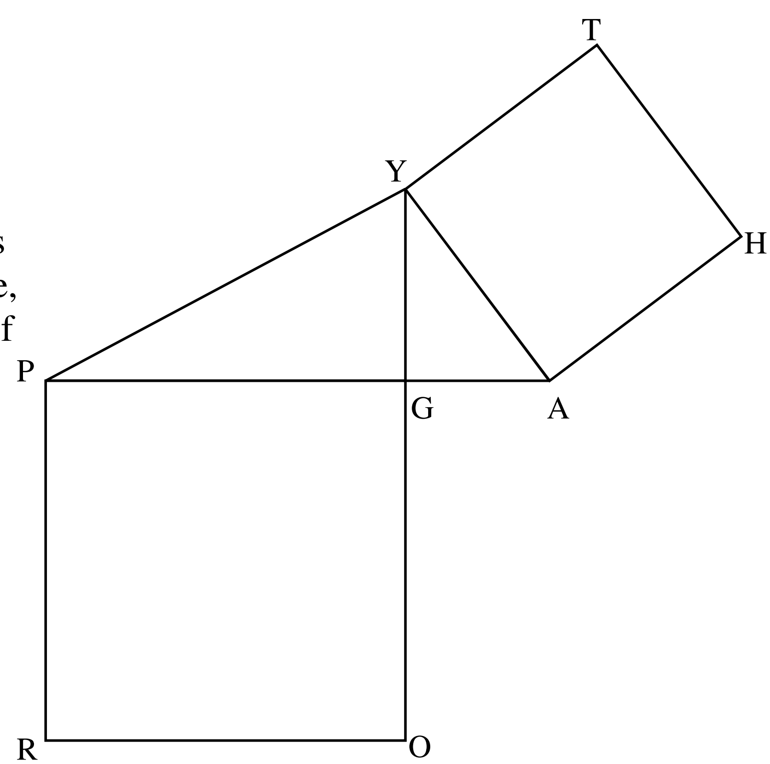
Answers
1. ________________
2. ________________
3. ________________

Category 2
Geometry
Meet #3, January, 2003

1. How many more diagonals can be drawn in a nonagon (a polygon with 9 sides) than in an septagon (a polygon with 7 sides)? (A diagonal of a polygon is a segment which connects any two non-consecutive vertices.)

2. An interior angle of a regular polygon is five times as great as an exterior angle on the same regular polygon. How many sides does the polygon have?

3. In the figure at right, segment YG measures 8 units, angle PGY is a right angle, square THAY has an area of 100 square units, angle OGA is a right angle, PGRO is a square, and triangle PGY has an area of 60 square units. How many units are in the perimeter of polygon PYTHAGOR?



Answers
1. ______________
2. ______________
3. ______________

Category 3
Number Theory
Meet #3, January, 2003

1. Express the base three number 21212 as a base ten number. In other words,

$$21212_{\text{base 3}} = \underline{\qquad}_{\text{ base 10}}$$

2. An astronomical unit (AU) is defined as the average distance from the sun to the earth and is about 9.3×10^7 miles. If Neptune is about 30 AU's from the sun, how many miles this? Express your answer in scientific notation without any further rounding.

3. How many digits are in the base-two expression of the base-eight number 671?

Answers
1. _______________
2. _______________
3. _______________

Category 4
Arithmetic
Meet #3, January, 2003

1. Evaluate the following expression.

$$19^0 + 7^4 \cdot 3^5 \cdot 7^{-3} \cdot 3^{-3} + 23^3 \cdot 23^{-3}$$

2. How many whole numbers are there between $\left(\frac{2}{3}\right)^{-2}$ and $\left(\frac{3}{4}\right)^{-4}$?

3. Evaluate the following expression.

$$\sqrt{\left(\sqrt{36 \cdot 49 \cdot 64} + \sqrt[3]{8 \cdot 27 \cdot 64} + \sqrt[6]{1}\right)}$$

Answers
1. _______________
2. _______________
3. _______________

Category 5
Algebra
Meet #3, January, 2003

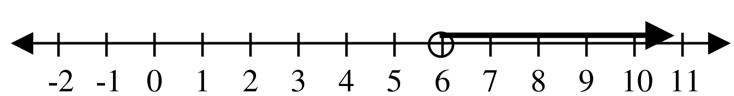
1. Given the following set of numbers, find the absolute value of the difference between the sum of the absolute values of the numbers and the absolute value of the sum of the numbers.

$$\{-10,\ 14,\ -5,\ 7,\ -3,\ -9,\ 1\}$$

2. For how many integer values of n is $\dfrac{28}{|n-3|}$ a positive integer?

3. For what value of a is the solution set of the inequality below given by the graph below?

$$3(2x-7)+9-11x<3a$$



Answers
1. _______________
2. _______________
3. _______________

Category 6
Team Questions
Meet #3, January, 2003

1. What number must be added to both the numerator and the denominator of $\frac{27}{72}$ to obtain a fraction whose value is equivalent to $\frac{4}{7}$?

2. A rectangular hallway measures 40 feet long and the diagonal of the hallway measures 41 feet. How many 12-inch by 12-inch square tiles did it take to tile the floor of the hallway?

3. If $8^x = 6784$, then what is the value of 8^{x-2}?

4. The base ten number 414 is a palindrome, that is it reads the same forward and backward. When 414 (base ten) is converted to another base it remains a palindrome. What base ten numeral names the least positive base for which this is true?

5. A certain number of consecutive odd whole numbers, starting with 1, has a sum of 1444. What is the average of these consecutive odd whole numbers?

Answers

1. ____________ $= A$
2. ____________ $= B$
3. ____________ $= C$
4. ____________ $= D$
5. ____________ $= E$
6. ______________

6. Using the values you obtained in questions 1 through 5, evaluate the following expression. Express your answer as a mixed number in lowest terms.

$$4C \cdot \frac{|A - E|}{B} + D$$

Solutions to Category 1
Mystery
Meet #3, January, 2003

Answers

1. Tuesday

2. 156

3. 6

1. Since Jill is walking to school, we can assume that June 2nd is a weekday. The only way that there can be five Sundays in the month is if June 2nd is a Monday. This makes June 1st, 8th, 15th, 22nd, and 29th Sundays. The 24th would be a **Tuesday**.

2. Let b be the price of the boots, s be the price of the skis, and g be the price of the bindings. Translating the second sentence to algebra, we have $b = 2s$ and $s = 3g$. Substituting $3g$ for s in the first equation, we get $b = 2(3g) = 6g$. The total cost of \$520 is $b + s + g$ or $6g + 3g + g$ or $10g$, which is to say that the total cost is ten times the cost of the bindings. The bindings must cost \$520 ÷ 10 or \$52. The skis cost three times as much as the bindings or 3 • \$52 or **\$156**.

3. Since all the balls weigh the same, it matters only how far out they are on each side of the beam.
On the left side of the beam, we have:
$2 \times 8 + 7 + 2 \times 5 + 2 + 1 = 16 + 7 + 10 + 2 + 1 = 36$.
On the right side of the beam, we have:
$3 \times 3 + 4 + 5 + 2 \times 6 = 9 + 4 + 5 + 12 = 30$.
If a single ball is placed under the **6** on the right side of the beam, the beam will balance.

Solutions to Category 2
Geometry
Meet #3, January, 2003

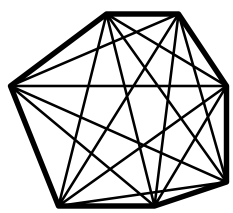
Answers

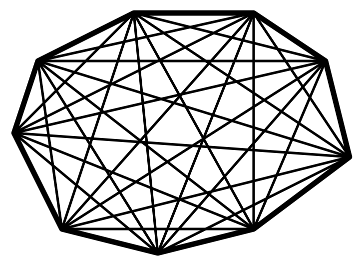
1. 13

2. 12

3. 98

1. The nonagon and the septagon need not be regular to determine the number of diagonals in each. From each vertex of the septagon, four diagonals can be drawn. If we multiply 4 • 7 = 28, we have counted each diagonal twice, so there must be 28 ÷ 2 = 14 diagonals. Similarly for the nonagon, we can draw six diagonals from each vertex, so there are 6 • 9 ÷ 2 = 27 diagonals. Thus there are 27 – 14 = **13** more diagonals on the nonagon than there are on the septagon.





2. An interior angle and an exterior angle of a regular polygon must be supplementary, that is they add up to 180 degrees. Since the interior angle is five times the exterior angle, we have $x + 5x = 180$. Solving this equation, we get $6x = 180$ and $x = 30$ degrees. In general, the exterior angle of a regular polygon is equal to 360 divided by the number of sides. Thus we have $360 \div n = 30$, and n must be **12**. The polygon has twelve sides.

3. Since square THAY has an area of 100 square units, we know its side length is 10 units. We now know YA is 10 and YG is 8. Using the Pythagorean Theorem, we can calculate the length GA. We have $x^2 + 8^2 = 10^2 \Rightarrow x^2 + 64 = 100 \Rightarrow x^2 = 36$, so GA is 6 units. Since the area of triangle PGY is 60 square units and the height is known to be 8 units, we can find the length of PG by solving $60 = \frac{1}{2} \cdot 8 \cdot x$ for x. PG is 15 units. Using the Pythagorean Theorem, we can calculate the length of PY: $8^2 + 15^2 = x^2 \Rightarrow 64 + 225 = x^2 \Rightarrow 289 = x^2$, so PY is 17 units. The perimeter of polygon PYTHAGOR is thus 17 + 10 + 10 + 10 + 6 + 15 + 15 + 15 = **98** units. *[Ed note: The original meet problem did not specify that PGRO is a square. Thanks to Danny Walker for pointing this out 12/2003.]*

Solutions to Category 3
Number Theory
Meet #3, January, 2003

Answers

1. 212

2. 2.79×10^9

3. 9

1. The place values in base three are powers of three. Thus the base-three number 21212 is equal to $2 \times 81 + 1 \times 27 + 2 \times 9 + 1 \times 3 + 2 = 162 + 27 + 18 + 3 + 2 =$ **212** in base ten.

2. Neptune's average distance from the sun is 30 AU's, or
$30 \times \left(9.3 \times 10^7\right) = 3 \times 10 \times 9.3 \times 10^7 = 3 \times 9.3 \times 10^8$
$= 27.9 \times 10^8 = \mathbf{2.79 \times 10^9}$ miles.

3. Converting the base-eight number 671 to base ten first, we get $671_{\text{base 8}} = 6 \times 64 + 7 \times 8 + 1 = 384 + 56 + 1$ $= 441_{\text{base 10}}$. Converting the base ten number 441 to base two, we subtract the largest power of two we can. Since $2^8 = 256$ is the largest power of two that is less than 441, we know already that the base two expression of the base eight number 671 will have nine (**9**) digits. (Remember that the place value of right-most place is $2^0 = 1$, also known as the units place.)

Solutions to Category 4
Arithmetic
Meet #3, January, 2003

Answers

1. 65

2. 1

3. 19

1. To evaluate the expression we must know that any number (other than zero) to the zero power is one. It also helps to know that we can add and subtract exponents when the bases are the same.

$$\begin{aligned} &19^0 + 7^4 \cdot 3^5 \cdot 7^{-3} \cdot 3^{-3} + 23^3 \cdot 23^{-3} \\ &= 19^0 + 7^{4-3} \cdot 3^{5-3} + 23^{3-3} \\ &= 19^0 + 7^1 \cdot 3^2 + 23^0 \\ &= 1 + 7 \cdot 9 + 1 \\ &= 1 + 63 + 1 \\ &= \mathbf{65}. \end{aligned}$$

2. Computing the value of each expression, we get:

$$\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{3^2}{2^2} = \frac{9}{4} = 2\frac{1}{4}, \text{ and}$$

$$\left(\frac{3}{4}\right)^{-4} = \left(\frac{4}{3}\right)^4 = \frac{4^4}{3^4} = \frac{256}{81} = 3\frac{13}{81}.$$

There is exactly one (**1**) whole number between $2\frac{1}{4}$ and $3\frac{13}{81}$, namely 3.

3. The square root of a product is equal to the product of the square roots. Thus, we can rewrite $\sqrt{36 \cdot 49 \cdot 64}$ as $\sqrt{36} \cdot \sqrt{49} \cdot \sqrt{64}$. Since each number is a perfect square, we have $6 \cdot 7 \cdot 8$ or 336. Similarly, we can rewrite $\sqrt[3]{8 \cdot 27 \cdot 64}$ as $\sqrt[3]{8} \cdot \sqrt[3]{27} \cdot \sqrt[3]{64}$. Each of these numbers is a perfect cube, so we have $2 \cdot 3 \cdot 4$ or 24. Finally, $\sqrt[6]{1}$ is equal to 1. We can now evaluate the original expression $\sqrt{\left(\sqrt{36 \cdot 49 \cdot 64} + \sqrt[3]{8 \cdot 27 \cdot 64} + \sqrt[6]{1}\right)}$ as $\sqrt{(336 + 24 + 1)} = \sqrt{361}$. Since $19^2 = 361$, we have $\sqrt{361} = \mathbf{19}$.

Solutions to Category 5
Algebra
Meet #3, January, 2003

Answers

1. 44

2. 12

3. -14

1. First let's find the sum of the absolute values of the numbers in the set: $|{-10}|+|14|+|{-5}|+|7|+|{-3}|+|{-9}|+|1|$ $=10+14+5+7+3+9+1=49$. Now let's find the absolute value of the sum of the numbers in the set: $|{-10}+14+{-5}+7+{-3}+{-9}+1|=|{-5}|=5$. Finally, the absolute value of the difference is $|49-5|=$ **44**.

2. For the expression $\frac{28}{|n-3|}$ to have an integer value, the denominator, $|n-3|$, must be a factor of 28. There are six positive factors of 28 (namely 1, 2, 4, 7, 14, and 28) and the corresponding six negative factors of 28. Thus there are twelve (**12**) different values of *n* for which the expression is an integer. It is not necessary to determine those values of *n*, but they are -25, -11, -4, -1, 1, 2, 4, 5, 7, 10, 17, and 31.

3. The graph shows $x > 6$. Distributing and combining like terms, we get:

$$3(2x-7)+9-11x<3a$$
$$6x-21+9-11x<3a$$
$$-5x-12<3a$$

Adding 12 to both sides we get:

$$-5x<3a+12$$

When we divide both sides of the equation by -5, the inequality switches direction, so we have:

$$x>\frac{3a+12}{-5}$$

Now we must find the value of *a* so that the expression on the right equals 6.

$$\frac{3a+12}{-5}=6$$
$$3a+12=-30$$
$$3a=-42$$
$$a=\mathbf{-14}$$

Solutions to Category 6
Team Questions
Meet #3, January, 2003

Answers

1. 33

2. 360

3. 106

4. 8

5. 38

6. $13\frac{8}{9}$

1. We must solve the equation $\frac{27+x}{72+x}=\frac{4}{7}$ for *x*.
The cross product give us a simpler equation:

$$7(27+x)=4(72+x)$$
$$189+7x=288+4x$$
$$3x=99$$
$$x=\mathbf{33}$$

A quick check shows that $\frac{27+33}{72+33}=\frac{60}{105}=\frac{4\times 15}{7\times 15}$.

2. We can find the width of the hallway using the Pythagorean Theorem. Solve the equation $x^2+40^2=41^2$ for *x*, we get:

$$x^2+1600=1681$$
$$x^2=81$$
$$x=\sqrt{81}=9$$

The hallway is 9 feet wide and 40 feet long for a total area of 360 square feet. Thus it took **360** 12-inch by 12-inch square tiles to tile the floor of the hallway.

3. The value of 8^{x-2} is equal to $8^x\cdot 8^{-2}$ or $\frac{8^x}{8^2}$. Since we know that $8^x=6784$, we must calculate $\frac{6784}{8^2}$ or $\frac{6784}{64}$, which comes to exactly **106**.

4. Since we are looking for the least base system for which the base ten number 414 is still a palindrome, we may as well convert 414 to base two and continue on to base three, etc. For base two, we get:
$414_{10} = 1\times 256 + 1\times 128 + 0\times 64 + 0\times 32 + 1\times 16 + 1\times 8 + 1\times 4 + 1\times 2 + 0\times 1$
$= 110011110_2$, which is not a palindrome.
For base three, we get:
$414 = 1\times 243 + 2\times 81 + 0\times 27 + 1\times 9 + 0\times 3 + 0\times 1 = 120100_3$, which is not a palindrome.
For base four, we get: $414 = 1\times 256 + 2\times 64 + 1\times 16 + 3\times 4 + 2\times 1 = 12132_4$, which is not a palindrome.
For base five, we get: $414 = 3\times 125 + 1\times 25 + 2\times 5 + 4\times 1 = 3124_5$, which is not a palindrome.
For base six, we get: $414 = 1\times 216 + 5\times 36 + 3\times 6 + 0\times 1 = 1530_6$, which is not a palindrome.
For base seven, we get: $414 = 1\times 343 + 1\times 49 + 3\times 7 + 1\times 1 = 1131_7$, which is not a palindrome.
Finally, for base eight, we get: $414 = 6\times 64 + 3\times 8 + 6\times 1 = 636_8$, which **is** a palindrome. Base eight (**8**) is the least base for which the expression of 414 is still a palindrome.

5. The sum of consecutive odd numbers (starting with 1) is always equal to a square number. For example, 1 + 3 = 4, 1 + 3 + 5 = 9, 1 + 3 + 5 + 7 = 16, etc. In each case the sum is equal to the square of the number of numbers added. In other words, the square root of the sum is the number of numbers added. We have to find the square root of 1444 without a calculator. $30^2 = 900$ and $40^2 = 1600$, so we should try the high thirties. The square of a number ending in 8 will end in 4, so we should try 38 and 38^2 is indeed 1444. Now, we have 38 numbers whose sum is 1444. The average will be 1444 ÷ 38 = **38**.

6. Substituting the values for *A* through *E* into the expression $4C\cdot\frac{|A-E|}{B}+D$,
we get: $4\cdot 106\cdot\frac{|33-38|}{360}+8 = 424\cdot\frac{5}{360}+8 = 424\cdot\frac{1}{72}+8 = \frac{424}{72}+8$
$= \frac{53\cdot 8}{9\cdot 8}+8 = \frac{53}{9}+8 = 5\frac{8}{9}+8 = \mathbf{13\frac{8}{9}}$.