Intermediate Mathematics League of Eastern Massachusetts

Meet #5
April, 2002

Category 1
Mystery
Meet #5, April, 2002

You may use a calculator today!

1. If seven horses drink 343 gallons of water in 7 days, how many gallons of water will 3 horses drink in 3 weeks?

2. Three people share a square sandwich. Unsure how to cut thirds, they cut the sandwich in four square pieces and they each eat one of the pieces. Then they cut the remaining piece into four square pieces and each eat one of these pieces. Then again they cut the remaining piece into four square pieces and each eat one of these pieces. This time they give what is left to the dog. How much less than a third of the sandwich does each person eat? Express your answer as a fraction in simplest form.

3. A farmer sold half his eggs and half an egg to his first customer. He then sold half of the remaining eggs and half an egg to his second customer. After that he sold half of the remaining eggs and half an egg to a third customer. Finally he sold his last five eggs to his fourth customer. If he did not break any eggs, how many eggs did he sell?

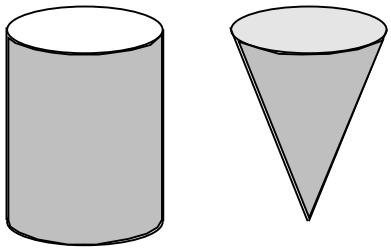
Answers

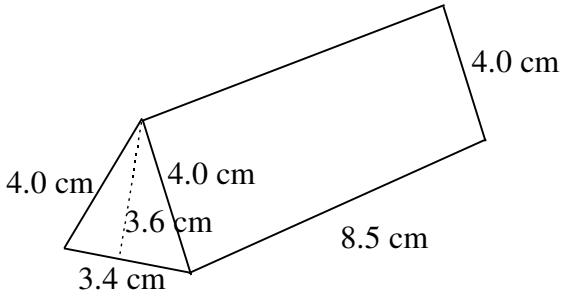
1. _____
2. _____
3. _____

Category 2
Geometry
Meet #5, April, 2002

You may use a calculator today!

1. A hollow cylinder whose diameter is 2 inches and whose height is 3 inches is filled with water. A solid cone also with a diameter of 2 inches and a height of 3 inches is pushed carefully into the cylinder, pointing down and causing some of the water to spill out. How many cubic inches are in the volume of the water that remains in the cylinder? Use 3.14 for Pi and express your answer as a decimal to the nearest hundredth.





2. Find the surface area of the triangular prism at left. Write your result as a decimal. (Note: The dotted line shows the altitude of the triangular base and measures 3.6 cm.)

3. A water tank has the shape of a cylinder with a hemisphere on top. The diameter of both the cylinder and the hemisphere is 40 feet. The height of the cylinder is 60 feet. If one gallon of paint covers 400 square feet, how many gallons will be needed to paint the surface of the water tank? Use 3.14 for π and round up to the next whole number of gallons.

Answers

1. _____
2. _____
3. _____

Category 3
Number Theory
Meet #5, April, 2002

You may use a calculator today!

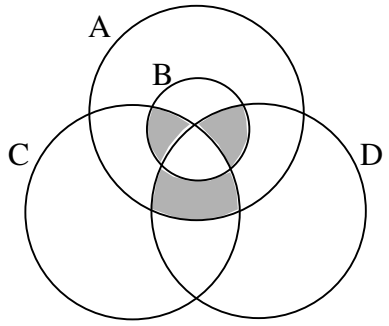
1. Given the following sets and the Venn diagram, find the sum of the numbers in the shaded regions.

$A = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9\}$

$B = \{0,\ 2,\ 4,\ 6,\ 8\}$

$C = \{-3,\ 0,\ 3,\ 6,\ 9,\ 12\}$

$D = \{-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3\}$



2. Set A is all the positive factors of 105. Set B is all the positive integers that are less than 22 and relatively prime to 22. How many elements are there in $A \cap B$? (Note: Two numbers are relatively prime if their GCF is 1.)

3. Of the 141 students in the 8th grade at Pleasant Town Middle School, 71 sing in the chorus, 84 play soccer, and 68 are girls. Exactly half of the girls who play soccer also sing in the chorus, but only one third of the boys who play soccer sing in the chorus. Of the 35 students in the chorus who don't play soccer, $\frac{4}{5}$ are boys. There are 24 girls who sing in the chorus and play soccer. How many girls in the 8th grade at Pleasant Town Middle School don't play soccer or sing in the chorus?

Answers

1. _____
2. _____
3. _____

Category 4
Arithmetic
Meet #5, April, 2002

You may use a calculator today!

1. Jerry's average for the first 6 quizzes in the quarter was 91%. He knows that there will be four more quizzes and that one of those quizzes is on a topic that he hates. If he aces the other three quizzes, what is the worst score he can get on one quiz and still maintain an average of 90%? (Note: The top score on a quiz is 100% and Jerry's teacher never rounds up, so he must get an average of exactly 90% or better to receive an A-.)

2. There are 5 red lollipops, 3 purple lollipops, and 4 green lollipops in a basket at the bank. Assuming that each of the flavors is equally likely to be picked, what is the probability that the next three lollipops taken from the basket will be green? Express your answer as a fraction in lowest terms.

3. A pair of number cubes each have the following powers of 2 on their six faces: 1, 2, 4, 8, 16, and 32. If the pair of number cubes is rolled and the numbers showing on top are added, what is the probability that the sum is a multiple of 3? Express your answer as a common fraction in simplest form.

Answers

1. _____
2. _____
3. _____

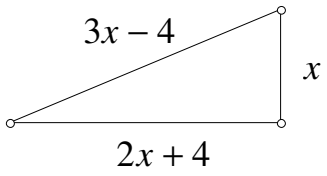
Category 5
Algebra
Meet #5, April, 2002

You may use a calculator today!

1. If two numbers have a sum of 30 and a product of 209, what is the positive difference between the two?

2. The roots of the equation $x^2 - 6x = b$ are 11 and a. Find the value of a.

3. Find the side length of a square that has the same area as the right triangle shown below. Round your answer to the nearest whole number of units.
(Hint: Use the Pythagorean Theorem and solve for x.)



Answers

1. _____
2. _____
3. _____

Category 6
Team Questions
Meet #5, April, 2002

You may use a calculator today!

Express all your answers to the nearest whole number.

1. A bathtub is $\frac{1}{4}$ full of water. When $3\frac{3}{4}$ gallons of water are added, the bathtub is $\frac{1}{3}$ full. How many gallons of water are needed to fill the bathtub the rest of the way?

2. A drill sells for $48.00. 80% of this price is the cost for the store and the rest is profit. How many dollars should the drill sell for so that 60% of the price is cost and the rest is profit?

3. The number 1365 is the product of 4 distinct primes. What is the positive difference between the mean and the median of these primes?

4. The longer base in a particular trapezoid is 2 more than twice the length of the shorter base. The height of this trapezoid is 2 less than twice the length of the shorter base. If the area of the trapezoid is 22 square units, how many units are in the length of the longer base?

5. On April 4th of the year 2016, Matilda's age will be $\frac{1}{31}$ of the year of her birth. How old is Matilda now, on April 4th of 2002?

Answers

1. _____ = A
2. _____ = B
3. _____ = C
4. _____ = D
5. _____ = E
6. _____

6. Using the values you obtained in questions 1 through 5, evaluate the following expression:

$$\sqrt[1]{A} + \sqrt[2]{E} + \sqrt[3]{D} + \sqrt[4]{4B} + \sqrt[5]{C} + \sqrt[6]{B} + \sqrt[7]{A + 2E}$$

Solutions to Category 1
Mystery
Meet #5, April, 2002

Answers

1. 441

2. $\frac{1}{192}$

3. 47

1. If seven horses drink 343 gallons of water, they have each drunk 49 gallons and the seven days is one week. Then 3 horses would drink $49 \times 3 = 147$ gallons in one week and $147 \times 3 = \mathbf{441}$ gallons in three weeks.

2. After the first cut, each person eats $\frac{1}{4}$ of the sandwich. The remaining quarter is cut if four pieces, creating sixteenths. Each person then eats $\frac{1}{16}$ of the sandwich. The remaining sixteenth is then cut into four pieces, creating sixty-fourths. Each person eats $\frac{1}{64}$ of the sandwich and the dog eats $\frac{1}{64}$ also. Each person has now consumed $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} = \frac{16}{64} + \frac{4}{64} + \frac{1}{64} = \frac{21}{64}$. (Another way to get this is to subtract the $\frac{1}{64}$ that the dog ate and divide the remaining $\frac{63}{64}$ by 3.) Finally, the question is how much less than one third is $\frac{21}{64}$? Subtracting, we get $\frac{1}{3} - \frac{21}{64} = \frac{64}{192} - \frac{63}{192} = \frac{\mathbf{1}}{\mathbf{192}}$.

3. To sell half of an odd number of eggs would require breaking one egg in half. The farmer avoided this inconvenience by selling half an egg more than half his eggs at each stage. Working backward, we know that the last 5 eggs represent half an egg less than half the eggs at the previous stage. He must have sold 6 to the third customer which means he had 11 before that. Again, the 11 eggs represent half an egg less than half the eggs at the previous stage; he must have sold 12 to the second customer which means he had 23 before that. Finally, the 23 eggs are half an egg less than half the eggs at the start. He must have sold 24 eggs to the first customer which means he started with **47** eggs and sold them all.

Solutions to Category 2
Geometry
Meet #5, April, 2002

Answers

1. 6.28

2. 109.14

3. 26

1. The volume of a cylinder is the area of the base times the height. The volume of a cone is $\frac{1}{3}$ the volume of a cylinder with the same diameter and the same height. This means that the volume of the water remaining in the cylinder is $\frac{2}{3}$ of the volume of the cylinder.

$$V_{\text{cylinder}} = \pi r^2 h = \pi \times 1^2 \times 3 = 3\pi$$

$$V_{\text{cone}} = \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3} \times \pi \times 1^2 \times 3 = \pi$$

$$V_{\text{cylinder}} - V_{\text{cone}} = 3\pi - \pi = 2\pi$$

Using 3.14 for Pi, we have:

$2\pi = 2 \times 3.14 =$ **6.28** cubic inches of water.

2. There are two identical triangular surfaces on the prism. Their area is $2 \times \frac{1}{2} bh = bh = 3.4 \times 3.6 = 12.24$ square centimeters. There are three rectangular surfaces on the prism, all of them with the length 8.5 cm. The sum of their areas is: $8.5 \times 4.0 + 8.5 \times 4.0 + 8.5 \times 3.4 = 34 + 34 + 28.9 = 96.9$. Alternatively, we can imagine unfolding the net of the prism and summing the widths of the three rectangles. The area would be the length times the sum of the widths or $8.5(4.0 + 4.0 + 3.4) = 8.5 \times 11.4 = 96.9$ square centimeters. Combining this area with the area of the triangles, we get a total surface area of **109.14** square centimeters.

3. The surface of the cylinder (top and bottom not included) is the circumference of the cylinder times the height. For the cylinder part of the water tower, we have a surface of $\pi Dh = 3.14 \times 40 \times 60 = 7536$ square feet. The surface of the hemisphere on top is one half the surface of a sphere or $\frac{1}{2} \times 4\pi r^2 = 2 \times 3.14 \times 20^2 = 2512$ square feet. This gives a total surface area of $7536 + 2512 = 10{,}048$ square feet. Since one gallon of paint covers 400 square feet, we will need to buy $10{,}048 \div 400 = 25.12$ or (rounding up to the next whole number of gallons) **26** gallons.

Solutions to Category 3
Number Theory
Meet #5, April, 2002

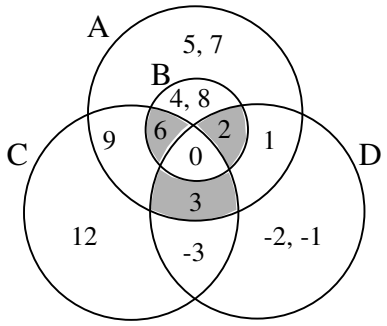
Answers

1. 11

2. 6

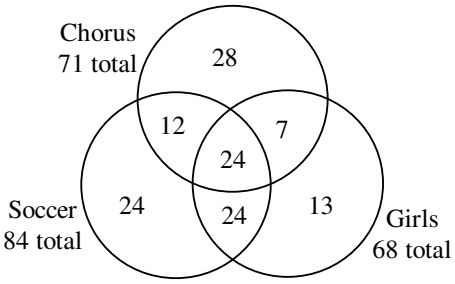
3. 13

1. The numbers from each set have been placed in their appropriate places in the diagram below. The sum of the numbers in the shaded regions is $6+2+3=\mathbf{11}$.



2. Set $A=\{1,\ 3,\ 5,\ 7,\ 15,\ 21,\ 35,\ 105\}$. Set $B=\{1,\ 3,\ 5,\ 7,\ 9,\ 13,\ 15,\ 17,\ 19,\ 21\}$. The intersection of these two sets is all the elements common to both, so $A\cap B=\{1,\ 3,\ 5,\ 7,\ 15,\ 21\}$. There are six (**6**) elements in $A\cap B$.

3. Using a Venn diagram, we can first place a 24 in the center for the 24 girls who sing in the chorus and play soccer. Next, the other half of the girls that play soccer (those who don't sing in the chorus) must also be 24. There must be 36 boys who play soccer, since $84-24-24=36$. One third of these boys, or 12, also sing in chorus and two thirds, or 24, don't sing in chorus. Four fifths of 35 is the 28 boys in chorus who don't play soccer. The remaining fifth of 35 is the 7 girls in chorus who don't play soccer. That leaves $68-24-24-7=\mathbf{13}$ girls who don't sing in the chorus and don't play soccer. (Incidentally, there must be 9 eighth grade boys who don't play soccer or sing in the chorus, since there are 141 eighth graders and only 132 people in the diagram.)



Solutions to Category 4
Arithmetic
Meet #5, April, 2002

Answers

1. 54 or 54%

2. $\frac{1}{55}$

3. $\frac{1}{2}$

1. If Jerry's average for the first 6 quizzes is 91%, then the sum of his scores is $91 \times 6 = 546$. To earn a 90% average on ten quizzes, he must have a sum of 900 percentage points. If he earns 100% on three of the quizzes he will have a sum of $546 + 300 = 846$. This means that he must earn at least **54%** for the quiz on the topic that he hates.

2. The probability that the next lollipop taken from the basket is green is 4 out of 12 or $\frac{4}{12}$. After that there are only 3 green lollipops among the 11 lollipops in the basket, so the probability is $\frac{3}{11}$. For the third lollipop to be green, the probability is $\frac{2}{10}$. The probability that three green lollipops will be taken in succession is the product of these separate probabilities, or $\frac{4}{12} \times \frac{3}{11} \times \frac{2}{10} = \frac{2}{110} = \mathbf{\frac{1}{55}}$.

3. Of the 36 possible rolls of the number cubes 18 give a sum that is a multiple of 3, as shown in the table. Thus the probability is $\frac{18}{36} = \mathbf{\frac{1}{2}}$.

+	1	2	4	8	16	32
1	2	**3**	5	**9**	17	**33**
2	**3**	4	**6**	10	**18**	34
4	5	**6**	8	**12**	20	**36**
8	**9**	10	**12**	16	**24**	40
16	17	**18**	20	**24**	32	**48**
32	**33**	34	**36**	40	**48**	64

Solutions to Category 5
Algebra
Meet #5, April, 2002

Answers

1. 8

2. -5

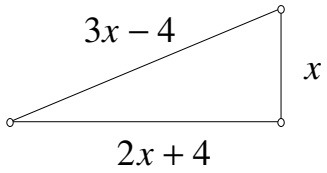
3. 11

1. Since there are many pairs of numbers with a sum of 30, it is helpful to narrow our search to the factors of 209. The only factors of 209 (besides 209 and 1 which give a sum of 210) are 11 and 19 and their sum is indeed 30. The positive difference between 19 and 11 is **8**.

2. To solve a quadratic our first step is to set it equal to zero. Thus $x^2 - 6x = b$ becomes $x^2 - 6x - b = 0$. Usually, our next step is to factor this quadratic into the product of two binomials of the form $(x \pm \quad)(x \pm \quad)$, where the signs and the blanks are determined by insight or by trial and error. In this case, we are told that one of the solutions is 11, which means that $(x - 11)$ is one of the binomials. The clue to the other solution is in the middle term of the quadratic. We must add $5x$ to $-11x$ to get $-6x$, so our product of two binomials is $(x - 11)(x + 5)$. This means that *a* (the other solution) is **-5**.

3. Before we can find the area of the triangle, we must use the Pythagorean Theorem to determine the unknown lengths.

$$x^2 + (2x + 4)^2 = (3x - 4)^2$$
$$x^2 + (4x^2 + 16x + 16) = 9x^2 - 24x + 16$$
$$5x^2 + 16x = 9x^2 - 24x$$
$$-4x^2 + 40x = 0$$
$$-4x(x - 10) = 0$$



The solution we want is $x = 10$, which is the shorter leg of the right triangle. The longer leg is $2x + 4 = 2 \times 10 + 4 = 24$ units. The area of the triangle is $\frac{1}{2} \times 10 \times 24 = 120$ square units. Since $11^2 = 121$, the closest whole number side length for a square with an area of 120 square units is **11** units.

Solutions to Category 6
Algebra
Meet #5, April, 2002

Answers

1. 30

2. 64

3. 1

4. 8

5. 49

6. 48

1. The $3\frac{3}{4}$ gallons of water that added represent $\frac{1}{12}$ of the volume of the bathtub, since $\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$. Thus the full capacity of the tub is $12\times 3\frac{3}{4}=12\times 3+12\times\frac{3}{4}=36+9=45$ gallons. The bathtub is already $\frac{1}{3}$ full, so we must add $\frac{2}{3}\times 45=$ **30** more gallons to fill it.

2. The cost of the drill is 80% of \$48.00 or $0.8\times\$48=\$38.40$. We want to find the price P such that $0.6\times P=38.40$. Dividing 38.40 by 0.6, we get **\$64.00**.

3. The four distinct prime factors of 1365 are 3, 5, 7, and 13. The average of these four numbers is $(3+5+7+13)\div 4=28\div 4=7$. The median is half way between 5 and 7, which is 6. The positive difference between 6 and 7 is **1**.

4. If we let x equal the shorter base, the longer base is $2x+2$ and the height is $2x-2$. We also know the area is 22 square units. Substituting the values of A, h, b_1, and b_2 into $A=\frac{1}{2}h(b_1+b_2)$, we get:

$$22=\tfrac{1}{2}(2x-2)\big(x+(2x+2)\big)=(x-1)(3x+2)$$
$$=3x^2+2x-3x-2=3x^2-x-2$$

Setting the equation $22=3x^2-x-2$ equal to zero to get: $3x^2-x-24=0$. Now we must factor this quadratic into the product of two binomials. With some effort we find that $(3x+8)(x-3)=0$ is the correct factorization. This product equals zero when one (or both) of its factors is zero, so we find the two solutions: $x=\frac{-8}{3}$ and $x=3$. The positive solution $x=3$ is the only solution that makes sense for a length. Now recall that we let x be the length of the shorter base. The longer base is $2x+2$ or $2\times 3+2=$ **8** units in length.

5. If we let x be Matilda's current age, then her birth year is $2002 - x$ and her age in 2016 will be $x + 14$. We are told that her age in 2016 is $\frac{1}{31}$ of her birth year. This translates to the equation: $x + 14 = \frac{1}{31}(2002 - x)$. If we multiply both sides of the equation by 31, we get the simpler equation: $31x + 434 = 2002 - x$. Solving for x, we get $32x = 1568$ and finally $x = 49$. Matilda is **49** years old now.

Alternatively, we could look for a birth year that is a multiple of 31. Some possible candidates are $1922 = 62 \times 31$, $1953 = 63 \times 31$, and $1984 = 64 \times 31$. A person born in 1922 will be 62 (which is $\frac{1}{31}$ of 1922) in 1984, not 2016. A person born in 1984 will be 64 (which is $\frac{1}{31}$ of 1984) in 2048, not 2016. But a person born in 1953 will be 63 (which is $\frac{1}{31}$ of 1953) in 2016. This person is currently (in 2002) $63 - 14 =$ **49**.

6. Substituting the values for A through E into the expression, we get:

$$\sqrt[1]{A} + \sqrt[2]{E} + \sqrt[3]{D} + \sqrt[4]{4B} + \sqrt[5]{C} + \sqrt[6]{B} + \sqrt[7]{A + 2E}$$

$$\sqrt[1]{30} + \sqrt[2]{49} + \sqrt[3]{8} + \sqrt[4]{4 \times 64} + \sqrt[5]{1} + \sqrt[6]{64} + \sqrt[7]{30 + 2 \times 49}$$

$$\sqrt[1]{30} + \sqrt[2]{49} + \sqrt[3]{8} + \sqrt[4]{256} + \sqrt[5]{1} + \sqrt[6]{64} + \sqrt[7]{128}$$

$$30 + 7 + 2 + 4 + 1 + 2 + 2 = \mathbf{48}$$