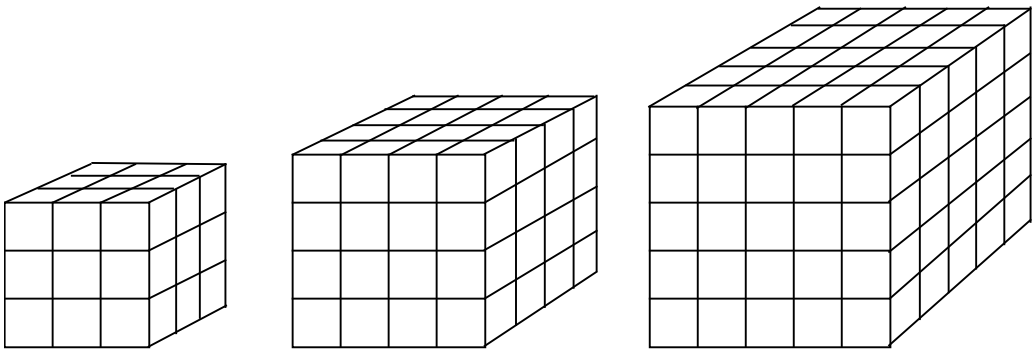
Intermediate Mathematics League of Eastern Massachusetts

Meet #4
February, 2002

Category 1
Mystery
Meet #4, February, 2002

You may use a calculator today!

1. Margie had a 3-by-3-by-3 cube, a 4-by-4-by-4 cube, and a 5-by-5-by-5 cube made from unit cubes that snap together. She disassembled the three cubes and combined all the unit cubes to make the biggest cube she could make. How many unit cubes did she have left over?



2. What is the value of n if 2^n is the least positive power of 2 that contains two consecutive 8's?

3. The number 6 can be written as the sum of distinct positive integers in the following four ways:

$$6,\ 5+1,\ 4+2,\ 3+2+1$$

(Note that 6 alone is to be considered a "sum" equaling 6, but $3+3$ is not allowed since the addends are not distinct.) How many ways can the number 11 be written as a sum of distinct positive integers?

Answers

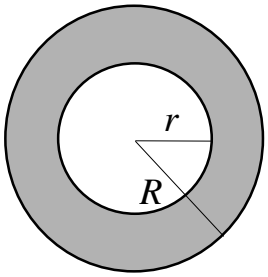
1. _____
2. _____
3. _____

Category 2
Geometry
Meet #4, February, 2002

You may use a calculator today!

1. A certain circle has a circumference of 18.84 centimeters. How many square centimeters are there in the area of the circle? Use $\pi = 3.14$ and express your result to the nearest hundredth of a square centimeter.

4. The two concentric circles shown at right were constructed so that the area of the shaded ring is twice the area of the inner circle. If the radius of the smaller circle is 1 centimeter, what is the radius of the larger circle? Give your answer to the nearest thousandth of a centimeter.



3. According to the ancient Egyptian Ahmes' papyrus, the area of a circle can be found by subtracting one-ninth of the diameter and then squaring the result. Find the area of a circle with a diameter of 10 feet by both the ancient Egyptian method and by our modern formula using 3.1416 for Pi. How many square feet are in the positive difference between these two calculations of the area of the circle? Round your answer to the nearest ten-thousandth.

Answers

1. _____
2. _____
3. _____

Category 3
Number Theory
Meet #4, February, 2002

You may use a calculator today!

1. Rhonda has a "loop" option on her CD player that plays the CD over again after the last track. When she sat down to start her homework one night, she put her favorite CD in the player, pressed play, and the first of the 17 tracks on the CD began to play. When Ronda finally finished her homework, she had listened to 57 songs completely and the 58^{th} song was playing. What was the number of the track that was playing when Rhonda finished her homework?

2. What is the sum of the first 85 terms of the following sequence?

5, 11, 17, 23, 29, 35,…

3. The following table is a part of a multiplication table in Modulo 18. Find the value of 11^{99} in Modulo 18.

x	1	5	7	11	13	17
1	1	5	7	11	13	17
5	5	7	17	1	11	13
7	7	17	13	5	1	11
11	11	1	5	13	17	7
13	13	11	1	17	7	5
17	17	13	11	7	5	1

Answers

1. _____
2. _____
3. _____

Category 4
Arithmetic
Meet #4, February, 2002

You may use a calculator today!

1. Jessica bought a telescope at 20% off. If the original price of the telescope was $485, how much money did she save by waiting for the sale?

2. When Shawn started his job at the furniture store in 1996 earning a salary of $31,500, he was promised a 4% annual raise. What was Shawn's salary for 2002? Round your result to the nearest dollar.

3. Rob loses 10% of his hair each year. In how many years will he have less than half the hair he has now?

Answers
1. _____
2. _____
3. _____

Category 5
Algebra
Meet #4, February, 2002

You may use a calculator today!

1. A dry-cleaner finds that a button pops off a shirt at a rate of about 1 button for every 175 shirts that are cleaned. If a thousand shirts are cleaned every week, how many buttons are expected to pop off in a year? Round your answer to the nearest hundred buttons.

2. There are a number of cows and chickens in the barnyard. The number of legs is 18 more than twice the number of heads. How many cows are in the barnyard?

3. Julie left the house at noon and rode her bicycle at an average speed of 15 miles per hour until she got a flat tire. She then walked home by the same route at an average speed of 3 miles per hour. If she arrived home at 4:00 PM, how far away was she when she got the flat tire?

Answers

1. _____
2. _____
3. _____

Category 6
Team Questions
Meet #4, February, 2002

You may use a calculator today!

1. How many three-digit numbers are there in base five?

2. Marco's age and his grandfather's age have the same two digits in the reverse order. The difference between their ages is seven elevenths of the sum of their ages. How old is Marco's grandfather?

3. The number 6 can be written as a sum of odd positive integers in the following four ways: $5+1,\ 3+3,\ 3+1+1+1,\ 1+1+1+1+1+1$. How many ways can the number 11 be written as a sum of odd positive integers?

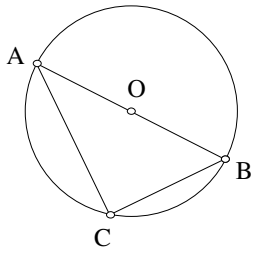
MATHLAND

36 @ [illegible] each

Total $ [illegible]8.5[illegible]

4. A coach bought 36 "MATH ROCKS!" buttons for his math team, but the receipt got wet and some of the digits were illegible. If the total was less than \$50 and the ones digit is clearly an 8 and the tenths digit is a clearly a 5, what is price of a single button? Express your answer in dollars to the nearest hundredth.

5. The area of circle O is 176.715 square units. If chord AC measures 12 units, how many square units are in the area of the triangle? Use $\pi = 3.1416$ and round your result to the nearest whole number.



Answers

1. _____ = A
2. _____ = B
3. _____ = C
4. _____ = D
5. _____ = E
6. _____

6. Using the values you obtained in questions 1 through 5, evaluate the following expression:

$$\frac{A^{\sqrt[4]{B}}}{\frac{C(E+B)}{B}} + AD$$

Solutions to Category 1
Mystery
Meet #4, February, 2002

Answers

1. 0

2. 19

3. 12

1. The 3-by-3-by-3 cube is made from $3\times 3\times 3 = 27$ unit cubes, the 4-by-4-by-4 cube is made from $4\times 4\times 4 = 64$ unit cubes, and the 5-by-5-by-5 cube is made from $5\times 5\times 5 = 125$ unit cubes. The total number of unit cubes is $27 + 64 + 125 = 216$. It so happens that $6\times 6\times 6 = 216$ also, so Margie can build a 6-by-6-by-6 cube and she will have zero (**0**) unit cubes left over.

2. Examining the powers of 2, we find that a double 8 does not arise until the 19th power of 2, so $n = \mathbf{19}$.

$2^1 = 2$	$2^6 = 64$	$2^{11} = 2048$	$2^{16} = 65{,}536$
$2^2 = 4$	$2^7 = 128$	$2^{12} = 4096$	$2^{17} = 131{,}072$
$2^3 = 8$	$2^8 = 256$	$2^{13} = 8192$	$2^{18} = 262{,}144$
$2^4 = 16$	$2^9 = 512$	$2^{14} = 16{,}384$	$2^{19} = 524{,}288$
$2^5 = 32$	$2^{10} = 1024$	$2^{15} = 32{,}768$	

3. The number 11 can be written as a sum of distinct positive integers in the following **12** ways:

As a single number: 11

As a sum of 2 numbers: $1+10,\ 2+9,\ 3+8,\ 4+7,\ 5+6$

As a sum of 3 numbers: $1+2+8,\ 1+3+7,\ 1+4+6,\ 2+3+6,\ 2+4+5$

As a sum of 4 numbers: $1+2+3+5$

Solutions to Category 2
Geometry
Meet #4, February, 2002

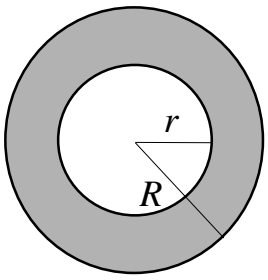
Answers

1. 28.26

2. 1.732

3. 0.4723

1. Given a diameter of a circle, its circumference can be found using the formula, $C = \pi D$. On the other hand, if the circumference is known, the diameter can be found be the formula $D = C \div \pi$. Our circle has a circumference of 18.84 cm, so its diameter must be $18.84 \div 3.14 = 6$ cm. A diameter of 6 cm means the radius measures 3 cm. The area of the circle is thus $A = \pi r^2 = 3.14 \times 3^2 = 3.14 \times 9 = \mathbf{28.26}$ square centimeters.



2. Since the shaded ring has *twice* the area of the inner circle, the circle of radius *R* must have exactly *three times* the area of the inner circle. This gives us the equation: $\pi R^2 = 3\pi r^2$. Substituting $r = 1$ and solving for *R*, we get $R^2 = 3$ or $R = \sqrt{3} \approx \mathbf{1.732}$ to the nearest thousandth of a centimeter.

3. By the ancient Egyptian method, the area of a circle with a 10-foot diameter would be calculated as follows:

$$D - \frac{1}{9}D = 10 - \frac{1}{9} \times 10 = \frac{90}{9} - \frac{10}{9} = \frac{80}{9} \Rightarrow \left(\frac{80}{9}\right)^2 = \frac{6400}{81} = 79.0123 \text{ sq. ft.}$$

Using the formula $A = \pi r^2$, with $\pi = 3.1416$ and a radius of 5 feet, we get: $A = 3.1416 \times 5^2 = 3.1416 \times 25 = 78.54$ square feet. The positive difference between these two calculations is $79.0123 - 78.5400 = \mathbf{0.4723}$, or a little less than one half of a square foot.

Solutions to Category 3
Number Theory
Meet #4, February, 2002

Answers

1. 7

2. 21,845

3. 17

1. Rhonda's CD played through three times completely, which is $17 \times 3 = 51$ songs. The 7^{th} track was playing when she finished her homework ($51 + 7 = 58$). The short cut is to divide 58 by 17 and the remainder of **7** is the answer.

2. In the sequence 5, 11, 17, 23, 29, 35,… the common difference is 6. This means that 84 sixes will be added to the first term to arrive at the 85^{th} term. Thus the 85^{th} term is $5 + 84 \times 6 = 509$. One way to find the sum of a sequence is to add the first and the last terms, multiply by the number of terms, and divide by two. Thus our sum is:

$$\frac{85(5+509)}{2} = \frac{85 \times 514}{2} = 85 \times 257 = \mathbf{21{,}845}.$$

3. Most calculators will show an overflow error if you try to compute 11^{99}, so you can't just calculate 11^{99} and then divide it by 18 to find the remainder. If we use the multiplication table that was provided to find the first few powers of 11, we see that the pattern shown at right emerges. There is a cycle of 6 values in the powers of 11 in Mod 18. We must determine what 99 equals in Mod 6 and then raise 11 to that power. Since $99 = 6 \times 16 + 3$, we have $11^{99}(\text{Mod}18) = 11^{3}(\text{Mod}18) = \mathbf{17}$.

$11^1 = 11,$
$11^2 = 13,$
$11^3 = 11^2 \times 11 = 13 \times 11 = 17$
$11^4 = 11^3 \times 11 = 17 \times 11 = 7$
$11^5 = 11^4 \times 11 = 7 \times 11 = 5$
$11^6 = 11^5 \times 11 = 5 \times 11 = 1$
$11^7 = 11^6 \times 11 = 1 \times 11 = 11$
$11^8 = 13$
$11^9 = 17$
etc.

Solutions to Category 4
Arithmetic
Meet #4, February, 2002

Answers

1. 97

2. 39,858

3. 7

1. Jessica saved 20% of \$485, which can be calculated as $0.2 \times 485 = 97$. She saved **\$97**.

2. In 1996, Shawn's salary was \$31,500. To obtain his salary for 1997, we could calculate 4% of \$31,500 and add this to \$31,500, or, to get the result more directly, we can multiply by 1.04 as follows: $\$31{,}500 \times 1.04 = \$32{,}760$. Continuing this process, we obtain the following results:
For 1998: $\$32{,}760 \times 1.04 = \$34{,}070.40$.
For 1999: $\$32{,}760 \times 1.04 = \$35{,}433.22$ (rounded)
For 2000: $\$35{,}433.22 \times 1.04 = \$36{,}850.55$.
For 2001: $\$36{,}850.55 \times 1.04 = \$38{,}324.57$.
For 2002: $\$38{,}324.57 \times 1.04 = \$39{,}857.55$.
Rounded to the nearest dollar this is **\$39,858**.
Alternatively, we can get the result more quickly using exponents as follows:
$31500 \times 1.04^6 = 39857.549$.

3. If Rob *loses* 10% of his hair each year, then (looking on the bright side) he *keeps* 90% of his hair each year. The question is how many times we have to multiply 0.9 by 0.9 before we get a result that is less than 0.5.
$0.9^1 = 0.9$, $0.9^2 = 0.81$, $0.9^3 = 0.729$, $0.9^4 = 0.6561$, $0.9^5 = 0.59049$, $0.9^6 = 0.531441$, $0.9^7 = 0.4782969$.
In **7** years, Rob will have less than half the hair he has now.

Solutions to Category 5
Algebra
Meet #4, February, 2002

Answers

1. 300

2. 9

3. 10

1. If the dry cleaner cleans 1000 shirts per week, then he cleans 52,000 shirts per year. If 1 button falls off for every 175, then we must solve the following proportion:

$$\frac{1}{175} = \frac{x}{52{,}000}$$

Thus $\frac{1 \times 52{,}000}{175} = x$, and $x = 297.14$, or about **300** buttons, to the nearest hundred, pop off of shirts every year.

2. This is a variation on a familiar Math League question involving the heads and legs of cows and chickens, two equations and two unknowns. The interesting twist this time is that we have no way of determining the number of chickens. If the number of legs were exactly twice the number of heads, then there would only be chickens in the barnyard. This means that the 18 extra legs all belong to cows. And since we know that every cow has two more legs than a chicken, there must be **9** cows in the barnyard.
Using Algebra, the statement in English becomes the following equation:
$L = 2H + 18$, where L is the number of legs and H is the number of heads.
For the number of heads, we can write:
$H = x + y$, where x is the number of cows and y is the number of chickens.
And, for the number of legs, we can write:
$L = 4x + 2y$, since cows have 4 legs and chickens have 2.
Substituting the second and third equations into the first, we get:
$4x + 2y = 2(x + y) + 18$, which becomes $4x + 2y = 2x + 2y + 18$, then $4x = 2x + 18$, then $2x = 18$, and finally $x = 9$, which is nine cows.

3. Since Julie rode her bicycle five times as fast as she walked, we know that it took five times as long to walk back than it did to ride out. These 5 parts and 1 part make 6 equal parts in time. Dividing the four hours into six equal parts, we get 40 minute parts. If she rode her bike for two-thirds of an

hour averaging 15 miles per hour, then she was 10 miles from home when she had the flat tire.

Solutions to Category 6
Team Questions
Meet #4, February, 2002

Answers

1. 100

2. 81

3. 12

4. 1.07

5. 54

6. 50,107

1. Since we can't have a leading zero in the left-most place of a three-digit number, there are only four possible digits that can occupy that place. There are five possible digits for the middle place and five possible digits for the right-most place. Thus there are $4 \times 5 \times 5 = 100$ possible three-digit numbers in base five.

2. Translating the English to Algebra, we get: $G - M = \frac{7}{11}(G + M)$, where M is Marco's age and G is his grandfather's age. We can rewrite this as the proportion:

$$\frac{G - M}{G + M} = \frac{7}{11}$$

It is helpful to notice that any two 2-digit numbers with reversed digits have a difference that is a multiple of 9 and a sum that is a multiple of 11. For the numerator in our proportion to be a multiple of 9, we can multiply numerator and denominator by 9 as follows:

$$\frac{G - M}{G + M} = \frac{7}{11} \times \frac{9}{9} = \frac{63}{99}$$

Since G and M are both limited to two-digits, the only way this proportion can work is if $G - M = 63$ and $G + M = 99$. Suppose now that Marco were the *same* age as his grandfather, then the sum of their ages would be $99 + 63 = 162$ and they would both be 81. Marco's grandfather is **81**.

3. The number 11 can be written as a sum of odd positive integers in the following **12** ways:

Starting with 11: 11

Starting with 9: $9+1+1$

Starting with 7: $7+3+1,\ 7+1+1+1+1$

Starting with 5: $5+5+1,\ 5+3+3,\ 5+3+1+1+1,\ 5+1+1+1+1+1+1$

Starting with 3: $3+3+3+1+1,\ 3+3+1+1+1+1+1,$
$3+1+1+1+1+1+1+1+1$

Starting with 1: $1+1+1+1+1+1+1+1+1+1+1$

4. The total cost (in cents) of 36 buttons must be a multiple of 36. We can test for divisibility by 36 using the tests for 4 and 9, since $4 \times 9 = 36$ and 4 and 9 are relatively prime. To be divisible by 4, the last two digits must be divisible by 4. There are two possibilities in the fifties: 52 or 56. To be divisible by 9, the digits of the number must add up to 9 or a multiple of 9. If we try $8 + 5 + 2$ for three of the digits, we get 15 and the remaining digit could be a 3. If we try $8 + 5 + 6$, we get 19 and the remaining digit would have to be an 8. The first possibility, \$38.52, is less than \$50. The second possibility, \$88.56, is not less than \$50. If 36 buttons cost \$38.52, then the price of a single button must be $38.52 \div 36 =$ **\$1.07**.

5. If the area of the circle is 176.715 square units, then we have:

$$A = \pi r^2 \Rightarrow 176.715 = 3.1416 \times r^2 \Rightarrow r^2 = 56.25 \Rightarrow r = 7.5$$

If the radius is 7.5 units, then the diameter is 15 units. It is important to notice that triangle ABC must be a right triangle, since AB is the diameter of a circle. This allows us to use the Pythagorean Theorem to find the length of segment CB. Let *b* equal the measure of segment CB.

$$a^2 + b^2 = c^2 \Rightarrow 12^2 + b^2 = 15^2 \Rightarrow 144 + b^2 = 225 \Rightarrow$$

$$b^2 = 225 - 144 \Rightarrow b^2 = 81 \Rightarrow b = 9$$

We can now find the area of the triangle since we know the lengths of both legs of a right triangle. $A = \frac{1}{2} \times 12 \times 9 =$ **54** square units.

6. Substituting the values for *A* through *E* into the expression, we get:

$$\frac{A^{\sqrt[4]{B}}}{\frac{C(E+B)}{B}} + AD = \frac{100^{\sqrt[4]{81}}}{\frac{12(54+81)}{81}} + 100 \times 1.07$$

$$= \frac{100^3}{\frac{12(135)}{81}} + 107 = \frac{1{,}000{,}000}{\frac{1620}{81}} + 107 = \frac{1{,}000{,}000}{20} + 107$$

$$= 50{,}000 + 107 = \mathbf{50{,}107}$$