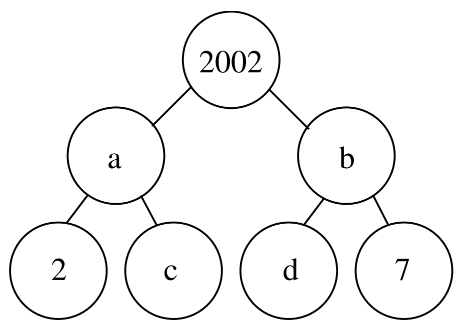
Intermediate Mathematics League of Eastern Massachusetts

Meet #3
January, 2002

Category 1
Mystery
Meet #3, January, 2002

1. How many positive whole numbers between 202 and 2002 are divisible by 4, 5, and 6?

2. What is the largest possible value of $a + b + c + d$ in the prime factorization tree shown?



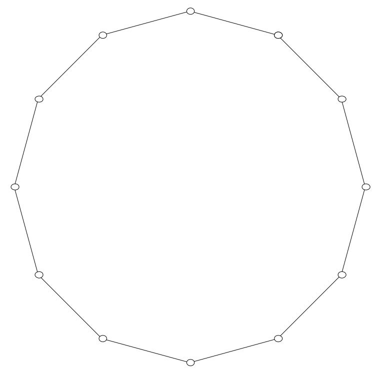
3. Freddie, Suzie, Josie, Lindie, and Paulie all chipped in to buy their friend Debbie a new guitar. Suzie gave $40 less than Josie. Josie gave $5 less than 3 times as much as Lindie. Freddie gave $60, which was twice as much as Suzie. How much must Paulie contribute if the guitar costs $250?

Answers
1. _______________
2. _______________
3. $______________

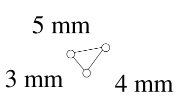
Category 2
Geometry
Meet #3, January, 2002

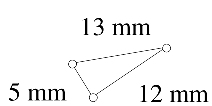
1. An exterior angle of a regular polygon measures 15 degrees. How many sides does the polygon have?

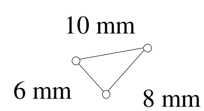
2. What is the maximum number of diagonals that can be drawn in the dodecagon shown at right?

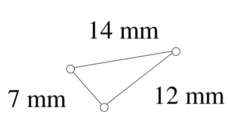


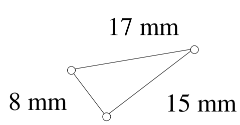
3. Given a right triangle, the Pythagorean Theorem tells us that the sum of the squares on the legs is equal to the square on the hypotenuse. The converse is also true, that is, if the sum of the squares on the shorter sides of a triangle is equal to the square on the longest side of the triangle, then the triangle is a right triangle. Write the letter(s) of those triangles below which are ***not*** right triangles.

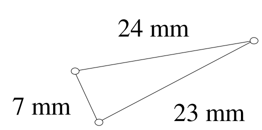
A. 

B. 

C. 

D. 

E. 

F. 

Answers
1. _______________
2. _______________
3. _______________

Category 3
Number Theory
Meet #3, January, 2002

1. What is the base 10 value of the base 6 number 1234?

2. Simplify the expression below. Write your result in scientific notation.

$$\frac{\left(9.1\times10^{10}\right)\left(4.5\times10^{-4}\right)}{\left(3.5\times10^{-6}\right)\left(3.9\times10^{16}\right)}$$

3. Find the base 5 product of the two base 5 numbers 3 and 4321.

Answers
1. _______________
2. _______________
3. _______________

Category 4
Arithmetic
Meet #3, January, 2002

1. Evaluate the following expression. Express your answer as a mixed number in simplest form.

$$3^3 + 2^2 + 1^1 + (-1)^{-1} + (-2)^{-2}$$

2. How many whole numbers are there between $\sqrt[3]{22}$ and $\sqrt[3]{2002}$?

3. Evaluate the following expression.

$$\sqrt[5]{72 \times \sqrt{9^3} \times \sqrt[3]{8^2}}$$

Answers
1. _______________
2. _______________
3. _______________

Category 5
Algebra
Meet #3, January, 2002

1. How many integer values of n satisfy the inequality $\left|\frac{15}{n}\right| > 4$?

2. Find the least value of x that makes the following equation true. Express your answer as a mixed number in lowest terms.

$$|5x - 17| = 40$$

3. Solve the following inequality for x. Write your solution in full, using x and the appropriate inequality symbol.

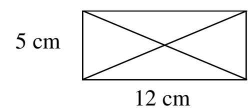
$$-4(7x + 5) \geq 64$$

Answers
1. _______________
2. _______________
3. _______________

Category 6
Team Questions
Meet #3, January, 2002

1. The average of two whole numbers is 18 and their product is 308. What is the positive difference between the two numbers?

2. How many centimeters are in the distance from a corner to the center of an 5 cm by 12 cm rectangle? Express your answer as a decimal to the nearest tenth.



3. If $2^x = 20$, then what is the value of 2^{x+3}?

4. Both $x = 8$ and $x = 18$ are solutions to one of the inequalities below, but $x = 0$ is *not* a solution to that same inequality. Which of the following could be the inequality? Write the number of the correct inequality.

1. $\lvert x-3\rvert > 10$	2. $\lvert x-3\rvert < 10$	3. $\lvert x-10\rvert > 3$	4. $\lvert x-10\rvert < 3$
5. $\lvert x+3\rvert > 10$	6. $\lvert x+3\rvert < 10$	7. $\lvert x+10\rvert > 3$	8. $\lvert x+10\rvert < 3$

5. Given that $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$, find the value of $a^2 + b^2 - c^2$ when $m = 7$ and $n = 6$.

Answers

1. __________ = A
2. __________ = B
3. __________ = C
4. __________ = D
5. __________ = E
6. ____________

6. Using the values you obtained in questions 1 through 5, evaluate the following expression:

$$\frac{\sqrt{C-2A}}{AB/(A+D)} + (ABCD)^E$$

Solutions to Category 1
Mystery
Meet #3, January, 2002

Answers

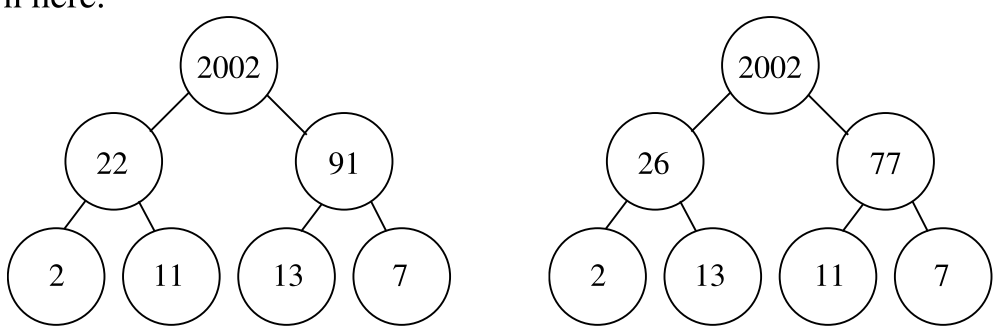
1. 30

2. 137

3. \$65

1. The least common multiple of 4, 5, and 6 is 60, so we need to know how many multiples of 60 there are between 202 and 2002. $4 \times 60 = 240$ is the least multiple of 60 in that range and $33 \times 60 = 1980$ is the greatest. From 4 to 33 is 30 multiples of 60 or **30** numbers that are divisible by 4,5, and 6.

2. The prime factorization of 2002 is $2 \times 7 \times 11 \times 13$. Since the 2 and the 7 are already written in the bottom row of the factor tree, we have just two ways to place the 11 and the 13. The two possible factor trees are shown shown here.



The first way gives us $a + b + c + d = 22 + 91 + 11 + 13 = 137$ and the second way gives us $a + b + c + d = 26 + 77 + 11 + 13 = 127$. **137** is the larger value.

3. Freddie gave \$60, Suzie gave \$30, Josie gave \$70, and Lindie gave \$25, for a sub-total of \$185. Thus Paulie must contribute **\$65** if the total is to reach \$250.

Solutions to Category 2
Geometry
Meet #3, January, 2002

Answers

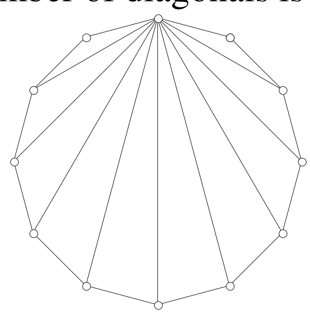
1. 24

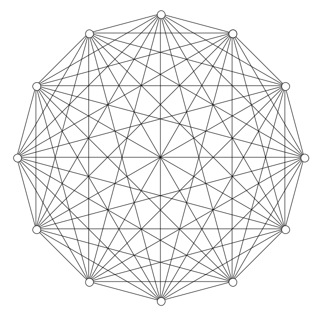
2. 54

3. D, F

1. If a person were to walk around the polygon, he would turn 15 degrees at each vertex and continue walking along each side. Since $15 \times 24 = 360$, he will turn 15 degrees at each of 24 vertices before returning to his starting point. A polygon with 24 vertices must have **24** sides.

2. Each vertex of a dodecagon can be connected to each of the 11 other vertices; 2 of those lines are sides of the dodecagon and the other 9 are diagonals. If we count 9 diagonals for each of the 12 vertices, we get $9 \times 12 = 108$, but we have counted each diagonal twice. Thus, the total number of diagonals is $108 \div 2 = \mathbf{54}$.





3. Testing the Pythagorean relationship for each triangle, we get:

A. $3^2 + 4^2 = 9 + 16 = 25$ and $5^2 = 25$, so it is a right triangle.

B. $5^2 + 12^2 = 25 + 144 = 169$ and $13^2 = 169$, so it is a right triangle.

C. $6^2 + 8^2 = 36 + 64 = 100$ and $10^2 = 100$, so it is a right triangle.

D. $7^2 + 12^2 = 49 + 144 = 193$ but $14^2 = 196$, so it's ***not*** a right triangle.

E. $8^2 + 15^2 = 64 + 225 = 289$ and $17^2 = 289$, so it is a right triangle.

F. $7^2 + 23^2 = 49 + 529 = 578$ but $24^2 = 576$, so it's ***not*** a right triangle.

Triangles D and F are not right triangles.

Solutions to Category 3
Number Theory
Meet #3, January, 2002

Answers

1. 310

2. 3×10^{-4}

3. 24013

1. Each place value in base 6 is a power of 6. Converting the base 6 number to base 10, we get: $1\times 216+2\times 36+3\times 6+4\times 1=216+72+18+4$ $=$ **310**.

2. The expression is more easily simplified if we deal with the decimal parts and the powers of 10 separately.

$$\frac{\left(9.1\times 10^{10}\right)\left(4.5\times 10^{-4}\right)}{\left(3.5\times 10^{-6}\right)\left(3.9\times 10^{16}\right)}=\frac{(9.1\times 4.5)\left(10^{10}\times 10^{-4}\right)}{(3.5\times 3.9)\left(10^{-6}\times 10^{16}\right)}$$

The decimal parts can be simplified as follows:

$$=\frac{(9.1\times 4.5)}{(3.5\times 3.9)}=\frac{(91\times 45)}{(35\times 39)}=\frac{(7\times 13\times 5\times 9)}{(5\times 7\times 3\times 13)}=3$$

And the powers of can be simplified as follows:

$$=\frac{\left(10^{10}\times 10^{-4}\right)}{\left(10^{-6}\times 10^{16}\right)}=\frac{10^{6}}{10^{10}}=\frac{1}{10^{4}}=10^{-4}$$

Thus our final simplified expression in scientific notation is: $\mathbf{3\times 10^{-4}}$.

3. We can multiply the numbers in base 5 as long as we remember to carry when we complete a group of 5. Working from the right, we have $3\times 1=3$ with nothing to carry. Then 3×2 (normally 6) is 1 group of 5 and 1 left over. Then 3×3 plus the 1 that was carried (normally 10) is 2 groups of 5 with zero left over. Finally, 3×4 plus the 2 that was carried (normally 14) is 2 groups of 5 and 4 left over. Our result is **24013** in base 5.

$$\begin{array}{r} {\scriptstyle 2\,1} \\ 4321 \\ \times\quad 3 \\ \hline 24013 \end{array}$$

Solutions to Category 4
Arithmetic
Meet #3, January, 2002

Answers

1. $31\frac{1}{4}$

2. 10

3. 6

1. The two terms with negative exponents are the tricky part of this problem.

$$(-1)^{-1} = \frac{1}{(-1)^1} = \frac{1}{-1} = -1, \text{ and}$$

$$(-2)^{-1} = \frac{1}{(-2)^2} = \frac{1}{4}$$

Evaluating the whole expression, we get:

$$3^3 + 2^2 + 1^1 + (-1)^{-1} + (-2)^{-2}$$

$$= 27 + 4 + 1 - 1 + \frac{1}{4} = \mathbf{31\frac{1}{4}}$$

2. If $x = \sqrt[3]{22}$, then $x^3 = 22$. We know that $2^3 = 8$ and $3^3 = 27$, so $\sqrt[3]{22}$ must be between 2 and 3. This means that 3 is the least whole number greater than $\sqrt[3]{22}$. Similarly, $\sqrt[3]{2002}$ must be between 12 and 13, since $12^3 = 1728$ and $13^3 = 2197$. This means that 12 is the greatest whole number less than $\sqrt[3]{2002}$. There are **10** whole numbers from 3 to 12 including 3 itself.

3. The fifth root of a number is easily computed when there turn out to be five of each prime factor. The expression evaluates as follows:

$$\sqrt[5]{72 \times \sqrt{9^3} \times \sqrt[3]{8^2}} = \sqrt[5]{72 \times \left(\sqrt{9}\right)^3 \times \left(\sqrt[3]{8}\right)^2}$$

$$= \sqrt[5]{72 \times (3)^3 \times (2)^2} = \sqrt[5]{8 \times 9 \times 3^3 \times 2^2}$$

$$= \sqrt[5]{2^3 \times 3^2 \times 3^3 \times 2^2} = \sqrt[5]{2^5 \times 3^5} = 2 \times 3 = \mathbf{6}.$$

Solutions to Category 5
Algebra
Meet #3, January, 2002

Answers

1. 6

2. $-4\frac{3}{5}$

3. $x \le -3$

1. There are **6** integer values of *n* that satisfy the inequality $\left|\frac{15}{n}\right| > 4$. They are:

$$n = -3,\ \left|\frac{15}{-3}\right| = |-5| = 5 > 4.$$

$$n = -2,\ \left|\frac{15}{-2}\right| = |-7.5| = 7.5 > 4.$$

$$n = -1,\ \left|\frac{15}{-1}\right| = |-15| = 15 > 4.$$

$$n = 1,\ \left|\frac{15}{1}\right| = |15| = 15 > 4.$$

$$n = 2,\ \left|\frac{15}{2}\right| = |7.5| = 7.5 > 4.$$

$$n = 3,\ \left|\frac{15}{3}\right| = |5| = 5 > 4.$$

2. We can expect two solutions to this equation; one for $5x - 17 = 40$ and another for $5x - 17 = -40$. Solving the first equation, we get: $5x = 57$ and then $x = 11\frac{2}{5}$. Solving the second equation, we get: $5x = -23$ and then $x = \mathbf{-4\frac{3}{5}}$. This second solution is the least possible solution.

3. As a first step, we can either divide both sides of the equation by -4 or distribute the factor of -4. The inequality is solved both ways below. In both cases, we must remember that the inequality changes direction when we divide both sides by a negative number.

$$\begin{array}{rcr} -4(7x+5) \ge 64 & \text{or} & -28x - 20 \ge 64 \\ (7x+5) \le -16 & & -28x \ge 84 \\ 7x \le -21 & & \boldsymbol{x \le -3} \\ \boldsymbol{x \le -3} & & \end{array}$$

Solutions to Category 6
Algebra
Meet #3, January, 2002

Answers

1. 8

2. 6.5

3. 160

4. 5

5. 0

6. 4

1. The prime factorization of 308 is $2^2 \times 7 \times 11$. Thus we have the following pairs of whole numbers with a product of 308: 2×154, 4×77, 7×44, 11×28, and 14×22. The two numbers of this last pair, 14×22, have the stated average of 18. The difference between 14 and 22 is **8**.
Jan04 note: Original question had average=17, but should be 18.

2. Using the Pythagorean Theorem, we can find the length of a diagonal of the rectangle.
$5^2 + 12^2 = 25 + 144 = 169$ and $\sqrt{169} = 13$
The two diagonals of a rectangle bisect each other, so the distance from any corner to the center of the rectangle is **6.5 cm**.

3. It is not necessary to determine the value of x to find the value of 2^{x+3}. From the laws of exponents, we know that $2^{x+3} = 2^x \times 2^3$. We are told that $2^x = 20$ and we know that $2^3 = 8$, so $2^x \times 2^3 = 20 \times 8 = \mathbf{160}$.

4. This problem can be solved by a process of elimination. Try $x = 8$ in each inequality and cross out those for which it is not a solution (1,3, 6, and 8). Now we try $x = 18$ in the remaining inequalities. This eliminates inequalities 2 and 4, leaving 5 and 7. We want the inequality for which $x = 0$ is *not* a solution. Inequality number **5** satisfies all three conditions. Alternatively, one could graph all eight equations and then check visually for the two solutions and the one non-solution.

5. Substituting $m = 7$ and $n = 6$ in the equations for *a, b,* and *c*, we find that:

$$a = m^2 - n^2 = 7^2 - 6^2 = 49 - 36 = 13$$

$$b = 2mn = 2 \times 7 \times 6 = 84, \text{ and}$$

$$c = m^2 + n^2 = 7^2 + 6^2 = 49 + 36 = 85$$

Now, substituting the values of *a, b,* and *c* into the final expression, we get:

$$a^2 + b^2 - c^2 = 13^2 + 84^2 - 85^2$$

$$= 169 + 7056 - 7225 = \mathbf{0}$$

6. Substituting the correct values for *A* through *E* gives:

$$\frac{\sqrt{C - 2A}}{AB/(A + D)} + (ABCD)^E = \frac{\sqrt{160 - 2 \times 8}}{8 \times 6.5/(8 + 5)} + (ABCD)^0$$

Notice that it is not necessary to find the product $ABCD$, since the exponent is zero and anything to the zero power is one. Thus we have:

$$\frac{\sqrt{160 - 16}}{52/(13)} + 1 = \frac{\sqrt{144}}{4} + 1 = \frac{12}{4} + 1 = 3 + 1 = \mathbf{4}.$$