Intermediate Mathematics League of Eastern Massachusetts

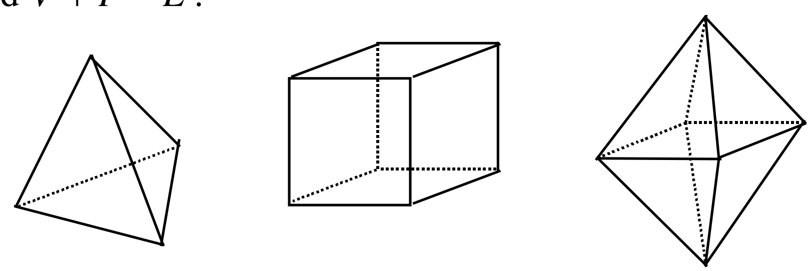
Meet # 5
April 2001

Category 1
Mystery
Meet #5, April 2001

You may use a calculator today!

1. What is the positive difference between the greatest and the least sums you can get from adding the squares of the digits of a 4-digit number?

2. Let V equal the total of the number of vertices on the tetrahedron, the hexahedron, and the octahedron shown below. Let F equal the total of the number of faces and let E equal the total of the number of edges on the three solids. Find $V + F - E$.



3. A passenger jet flies non-stop all the way around the world 5 miles above the equator. A military jet "shadows" the passenger jet on the same flight, but at an altitude of 6 miles. If both jets fly in perfect circles, how much farther does the military jet fly? You may use 3960 miles as the radius of earth at the equator and 3.14 for the value of pi. Give your answer to the nearest tenth of a mile.

Answers
1. _______________
2. _______________
3. _______________

Category 2
Geometry
Meet #5, April 2001

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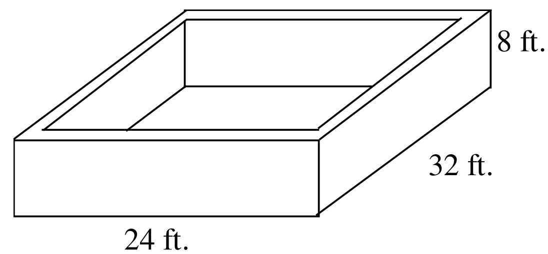
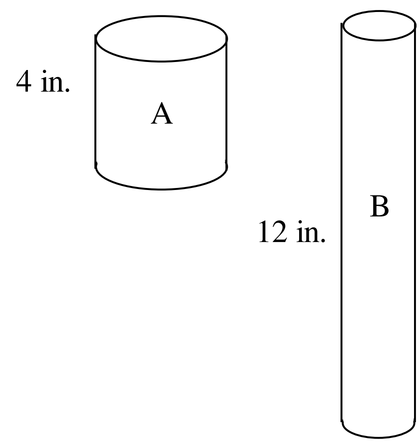
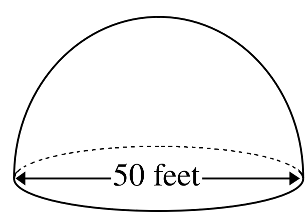
1. How many 8-inch by 8-inch by 16-inch cement blocks will be required to construct a foundation that is 32 feet long, 24 feet wide, and 8 feet tall? (Each wall is 8 inches thick.)



2. The label from each of the two cans shown here is a 4-inch by 12-inch rectangle which exactly covers the rounded surface of each can, but not the top and bottom. If the height of can A is 4 inches and the height of can B is 12 inches, what is the ratio of the volume of can A to the volume of can B? Express your answer as a ratio in the form *a*:*b*, where the greatest common factor of *a* and *b* is 1.



3. A exterior surface of a hemispherical capital dome is to be covered with gold leaf. If the diameter of the dome is 50 feet, how many square feet of gold leaf will be required to cover the dome? Use 3.14 for Pi and give your answer to the nearest whole number of square feet.



Answers
1. _______________
2. _______________
3. _______________

Category 3
Number Theory
Meet #5, April 2001

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1. In set notation, $|A|$ means "the number of elements in set *A*". For example, if $A = \{8,9,10\}$, then $|A| = 3$. For this problem, set *A* is all the multiples of 2 between 0 and 100, set *B* is all the multiples of 3 between 0 and 100, set *C* is all the perfect squares between 0 and 100, and set *D* is all the perfect cubes between 0 and 100. Find $|(A \cup B) \cap (C \cup D)|$.

2. Of the 25 students in Linda's class, 13 went skiing over vacation, 15 saw a movie, and 9 worked on math problems. Three students did all of these things; three students did none of these things; three students skied and did math problems, but did not see a movie; three students did math problems and saw a movie, but did not ski; and three students saw a movie and skied, but did not do math problems. How many of the students in Linda's class did exactly one of these activities over vacation?

3. During the holidays, the 43 students on math team had a party and the 58 students in chorus had a party. Nine students got to go to both parties! Jill decided to have a party for the students in math team *who are not in chorus* and for the students in chorus *who are not on math team*. If every student who qualifies comes, how many students will be at the party?

Answers
1. _______________
2. _______________
3. _______________

Category 4
Arithmetic
Meet #5, April 2001

You may use a calculator today!

1. Two number cubes are rolled. Each number cube has the whole numbers 1 through 6 on its six faces. What is the probability that the product of the two numbers on the top of the cubes is a multiple of six? Express your answer as a fraction in lowest terms.

2. There are 6 butterscotch and 8 peppermint candies in a jar. What is the probability of drawing first a butterscotch and then a peppermint candy from the jar? Express your answer as a fraction in lowest terms.

3. Albert had an average of 88% on six quizzes, all of which were weighted equally. If 100% is the highest score he could have received on any of the quizzes, what is the lowest possible score he could have received on one of the quizzes? Express your answer to the nearest percent.

Answers
1. _______________
2. _______________
3. _______________

Category 5
Algebra
Meet #5, April 2001

You may use a calculator today!

1. There are two solutions to the equation $x^2 + 6x = 91$. If these two solutions were plotted on the real number line, what would the distance between them be?

2. A rectangular room has a perimeter of 60 feet and an area of 216 square feet. How many feet longer is this rectangle than it is wide?

3. A rational number greater than one and its reciprocal have a sum of $2\frac{1}{6}$. What is this number? Express your answer as an improper fraction in lowest terms.

Answers
1. _______________
2. _______________
3. _______________

Category 6
Team Questions
Meet #5, April 2001

You may use a calculator today!

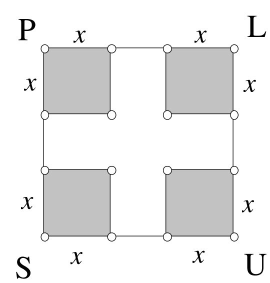
1. Two numbers are chosen at random, without replacement, from the set $\{1, 2, 3, 4, 5, 6\}$ and used to form a fraction that is less than one. What is the probability that the fraction cannot be reduced? Express your answer as a fraction in lowest terms.

2. What is the least positive multiple of 9 that leaves a remainder of 5 when divided by both 7 and 8?

3. Nick averaged 3 kilometers per hour going up the hill on snowshoes and 21 kilometers per hour going down on a sled. What was Nick's average speed in kilometers per hour for the entire trip? Express your answer as a decimal to the nearest hundredth.

4. Heron's formula calculates the area of a triangle from the lengths of its sides. The formula is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where *a, b,* and *c* are the lengths of the sides and *s* is the semiperimeter (half the perimeter) of the triangle. How many square centimeters are in the area of a triangle whose sides are 13 cm, 14 cm, and 15 cm?

5. If the side length of square PLUS is 8 cm, find the measure of *x* so that the total area of the shaded regions is equal to the area of the white region. Express your answer as a decimal to the nearest tenth of a centimeter.



Answers	
1. __________	= A
2. __________	= B
3. __________	= C
4. __________	= D
5. __________	= E
6. ____________	

6. Evaluate the following expression, using the answers to questions 1 through 5 as the values of *A, B, C, D,* and *E*, respectively:

$$\sqrt{\left(\frac{11}{A} + \frac{10B}{9} + \frac{10C}{2.5} + \frac{D}{E}\right)}$$

Solutions to Category 1
Mystery
Meet #5, April 2001

Answers

1. 323

2. 6

3. 6.3

1. The greatest sum you can get from the squares of the digits of a 4-digit number comes from the number 9999 and is $81+81+81+81=324$. The least sum of the squares of the digits of a 4-digit number comes from the number 1000 and is 1. The positive difference between 324 and 1 is 323.

2. Students must carefully count the total number of vertices, faces, and edges on the tetrahedron, the hexahedron, and the octahedron. We have: $V=4+8+6=18$, $F=4+6+8=18$, and $E=6+12+12=30$. Therefore, $V+F-E$ is $18+18-30=6$.

3. It is not necessary to know the radius of the earth to answer this question. The passenger jet will fly $2\pi(r+5)$ or $2\pi r+10\pi$ miles, whereas the military jet will fly $2\pi(r+6)$ or $2\pi r+12\pi$ miles. By subtracting $(2\pi r+10\pi)$ from $(2\pi r+12\pi)$, we find that the military jet will fly 2π miles or about 6.3 miles farther than the passenger jet.

Solutions to Category 2
Geometry
Meet #5, April 2001

Answers

1. 984

2. 3:1

3. 3925

1. The foundation wall is 8 feet, or 96 inches, tall all the way around and will require 12 layers of cement blocks that are 8 inches tall. Each of the sides that are 32 feet long will require $32 \times 12 \div 16$ or 24 blocks that are 16 inches long on each layer. Similarly, the sides that are 24 feet long will require $24 \times 12 \div 16$ or 18 blocks on each layer. Watch out! The half-block in each corner on each layer is shared by two walls. If we count our way around one level of the foundation—$24 + 18 + 24 + 18 = 84$ blocks—we have overcounted by 4 halves or 2 blocks. We will need only 82 blocks on each of 12 layers of block, for a grand total of 984 blocks.

2. We know from the dimensions of the label that can A must have a circumference of 12 inches. From this we can find that the radius of can A must be $\frac{12}{2\pi} = \frac{6}{\pi}$. Similarly, we can find the radius of can B, which is $\frac{4}{2\pi} = \frac{2}{\pi}$. Using the volume formula $V = Area_{base} \times height = \pi r^2 h$, we find:

Can A: $V = \pi\left(\frac{6}{\pi}\right)^2 \times 4 = \pi \times \frac{36}{\pi^2} \times 4 = \frac{144}{\pi}$

Can B: $V = \pi\left(\frac{2}{\pi}\right)^2 \times 12 = \pi \times \frac{4}{\pi^2} \times 12 = \frac{48}{\pi}$

The ratio $\frac{144}{\pi} : \frac{48}{\pi}$ can be simplified to 3:1.

3. The formula for the surface area of a sphere is $4\pi r^2$, so the surface of a hemisphere will be half of that. Our dome has a diameter of 50 and a radius of 25, so $2 \times 3.14 \times 25^2 = 6.28 \times 625 =$ 3925 square feet.

Solutions to Category 3
Number Theory
Meet #5, April 2001

Answers

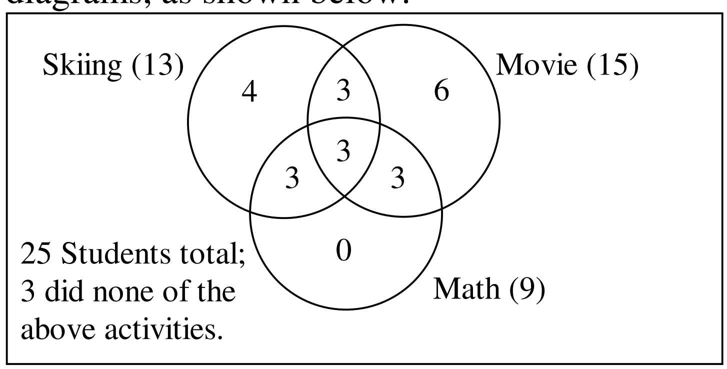
1. 8

2. 10

3. 83

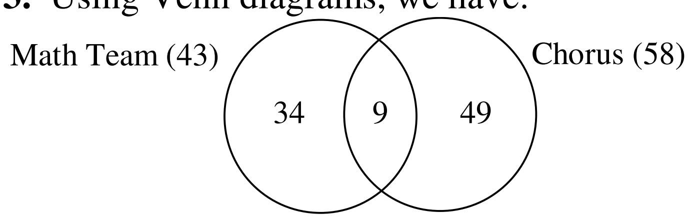
1. The elements of $(A \cup B) \cap (C \cup D)$ are either perfect squares or perfect cubes that are either multiples of 2 or multiples of 3, between 0 and 100. The resulting set is $\{4, 8, 9, 16, 27, 36, 64, 81\}$, which has 8 elements, thus $|(A \cup B) \cap (C \cup D)| = 8$.

2. This problem is best solved using Venn diagrams, as shown below.



There were 4 + 6 + 0 or 10 students who did exactly one of the three activities over vacation.

3. Using Venn diagrams, we have:



There are 34 + 49 or 83 students who are on the math team or in chorus but not both. Another way to arrive at this result is to add the numbers of students in math team and chorus (43 + 58 = 101) and subtract *twice* the number who do both ($101 - 2 \times 9 = 101 - 18 = 83$).

Solutions to Category 4
Arithmetic
Meet #5, April 2001

Answers

1. $\frac{5}{12}$

2. $\frac{24}{91}$

3. 28%

1. A multiplication chart is helpful in answering this question.

X	1	2	3	4	5	6
1	1	2	3	4	5	**6**
2	2	4	**6**	8	10	**12**
3	3	**6**	9	**12**	15	**18**
4	4	8	**12**	16	20	**24**
5	5	10	15	20	25	**30**
6	**6**	**12**	**18**	**24**	**30**	**36**

Out of 36 possible outcomes, 15 are multiples of 6, hence $\frac{15}{36}$ or $\frac{5}{12}$ is the probability that the product of the number cubes is a multiple of 6.

2. The first candy drawn from the jar is to be a butterscotch candy. There are 6 butterscotch out of 14 candies total, so the probability of drawing a butterscotch is $\frac{6}{14}$. Now we must choose one of the 8 peppermint candies from the 13 remaining candies in the jar. The probability of this happening is $\frac{8}{13}$. The probability of these two events happening in succession is the product of their probabilities:

$$\frac{6}{14} \times \frac{8}{13} = \frac{48}{182} = \frac{24}{91}$$

3. If Albert averaged 88% on six quizzes, then the sum of his six unknown scores is $6 \times 88 = 528$. If he received 100% on five of the quizzes, 28% is the lowest score he could have received on the sixth quiz.

Solutions to Category 5
Algebra
Meet #5, April 2001

Answers

1. 20

2. 6

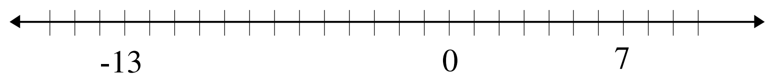
3. $\frac{3}{2}$

1. The equation $x^2 + 6x = 91$ can be set equal to zero and factored as follows:

$$x^2 + 6x - 91 = 0$$

$$(x-7)(x+13) = 0, \quad x = 7 \text{ or } {}^{-}13$$

Plotting these two solutions on a number line, we find them to be 20 units apart.



2. We need two numbers whose product is 216 and whose sum is 30 (half the perimeter). Trial and error, using factors of 216, is the best way to solve this problem. We might try $8 \times 27 = 216$, but $8 + 27 = 35$; or $9 \times 24 = 216$, but $9 + 24 = 33$. By combining the prime factors in different ways, eventually we find $12 \times 18 = 216$ and $12 + 18 = 30$. The room is 18 feet by 12 feet, which is 6 feet longer than it is wide.

Notice that algebra is no help for this problem. We can write the equations $lw = 216$ and $l + w = 30$, substitute $l = 30 - w$ into the first equation, and get $(30 - w)w = 216$. Rearranging things, we end up looking at the equation $w^2 - 30w + 216 = 0$. We could use the quadratic formula, but it's often quicker to factor this into the form $(w - \quad)(l - \quad)$=0. To fill in those blanks, we need two numbers whose product is 216 and whose sum is 30. We're back where we started and trial and error is the best way!

3. A number and its reciprocal with a sum of $2\frac{1}{6}$ may be represented algebraically as follows: $x + \frac{1}{x} = 2\frac{1}{6}$ or $x + \frac{1}{x} = \frac{13}{6}$. Multiplying through by 6x, we get: $6x^2 + 6 = 13x$. Setting this equal to zero, gives: $6x^2 - 13x + 6 = 0$, which factors to: $(2x-3)(3x-2) = 0$, so $x = \frac{2}{3}$ or $\frac{3}{2}$. We want the solution that is greater than 1 or $\frac{3}{2}$. This may also be solved by trial and error.

Solutions to Category 6

Team Questions

Meet #5, April 2001

Answers

1. $\frac{11}{15}$

2. 117

3. 5.25

4. 84

5. 2.8

6. 14

1. Listing all the possible fractions less than one that can be formed from the set, we have:
$\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$; $\frac{2}{3}$, $\frac{2}{4}$, $\frac{2}{5}$, $\frac{2}{6}$; $\frac{3}{4}$, $\frac{3}{5}$, $\frac{3}{6}$; $\frac{4}{5}$, $\frac{4}{6}$; $\frac{5}{6}$
Of these 15 fractions, 11 cannot be reduced.

2. A number that leaves a remainder of 5 when divided by both 7 and 8 must be 5 more than a multiple of 7×8 or 56. We must look for a multiple of 9 that is 5 more than a multiple of 56. Our first try, $1 \times 56 + 5 = 61$, is not a multiple of 9, but our second try, $2 \times 56 + 5 = 117$, works: $117 = 13 \times 9$.

3. It is not necessary to know how big a hill Nick was sledding on. His speed going down the hill on a sled was 7 times as fast as his speed going up the hill on snowshoes. In other words, it took him 7 times as long to go up the hill as it did to sled down the hill. It terms of time, we have a ratio of 7 parts to 1 part, for 8 equal parts total. He spends 7 units of time at 3 kph and 1 unit of time at 21 kph. His average speed can be found as follows:

$$\frac{7 \times 3 + 1 \times 21}{8} = \frac{42}{8} = 5.25 \text{ kph.}$$

Another way to solve this is to introduce a convenient distance, such as 21 km. If the distance were 21 km, then it would take Nick 7 hours to go up the hill at 3 kph and 1 hour to come down at 21 kph. That's a total of 42 km traveled in 8 hours, which is an average speed of 5.25 kph.

4. A triangle with side lengths 13, 14, and 15 has a perimeter of $13+14+15=42$ and a semi-perimeter of 21. Using Heron's formula, we get:

$$A=\sqrt{21(21-13)(21-14)(21-15)}$$

$$=\sqrt{21\times 8\times 7\times 6}=\sqrt{3^2\times 4^2\times 7^2}=3\times 4\times 7=84$$

The triangle has an area of 84 square centimeters.

5. The area of square PLUS is 64 square centimeters. We have four shaded squares, each with an area of x^2 square cm, for a total of $4x^2$ square cm. If the area of the shaded squares and the white region is to be equal, then the $4x^2$ square cm is $\frac{1}{2}$ of square PLUS or 32 square cm. Solving the equation $4x^2=32$ for x yields $x=\sqrt{8}\approx 2.8$ to the nearest tenth of an centimeter.

6. Substituting for A through E gives:

$$\sqrt{\left(\frac{11}{A}+\frac{10B}{9}+\frac{10C}{2.5}+\frac{D}{E}\right)}=$$

$$\sqrt{\left(\frac{11}{\frac{11}{15}}+\frac{10\times 117}{9}+\frac{10\times 5.25}{2.5}+\frac{84}{2.8}\right)}$$

$$\sqrt{(15+130+21+30)}=\sqrt{196}=14$$