Meet # 4
March 2001

Intermediate Mathematics League of Eastern Massachusetts

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Category 1
Mystery
Meet #4, March 2001

You may use a calculator today!

1. What fraction is $37\frac{1}{2}\%$ greater than $\frac{2}{3}$? Express your answer in lowest terms.

2. The harmonic mean of two numbers is the reciprocal of the mean of the reciprocals of the two numbers. It can also be found by dividing two times the product of the two numbers by the sum of the two numbers. What is the harmonic mean of 13 and 18? Round your answer to the nearest tenth.

5. For what whole number value of n is the following equation true?

$$7^n + 24^n = 25^n$$

Answers
1. _____
2. _____
3. _____

Solutions to Category 1
Mystery
Meet #4, March 2001

Answers

1. $\frac{11}{12}$

2. 15.1

3. 2

1. Since the fraction we want is $37\frac{1}{2}\%$ *greater than* $\frac{2}{3}$, it must be $137\frac{1}{2}\%$ *of* $\frac{2}{3}$. With a calculator, one could simply multiply 1.375 by $\frac{2}{3}$ and get 0.916666667, then, using the fraction-to-decimal key, obtain the fraction $\frac{11}{12}$.
Alternatively, a student might realize that $37\frac{1}{2}\%$ is equivalent to the fraction $\frac{3}{8}$ and $\frac{3}{8}\times\frac{2}{3}=\frac{1}{4}$.
The final step is to add as follows:
$\frac{2}{3}+\frac{1}{4}=\frac{8}{12}+\frac{3}{12}=\frac{11}{12}$.

2. The problem gives two ways to obtain the harmonic mean of 13 and18. The first method is rather difficult:

$$\frac{1}{\frac{\left(\frac{1}{13}+\frac{1}{18}\right)}{2}}=\frac{1}{\frac{\left(\frac{18+13}{234}\right)}{2}}=\frac{1}{\frac{\left(\frac{31}{234}\right)}{2}}=\frac{1}{\left(\frac{31}{468}\right)}$$

$$=\frac{468}{31}\approx 15.1$$

The second method is much easier:

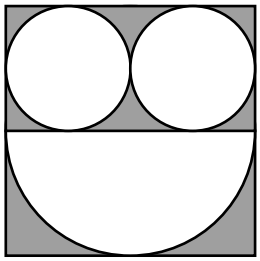
$$\frac{2\times 13\times 18}{13+18}=\frac{468}{31}\approx 15.1$$

3. Students who have heard of Fermat's Last Theorem will know that the only possible integer solution for this type of problem is $n=2$. Other students will have to try different values of *n*. Since $n=0$ and $n=1$ can be ruled out quickly, they should soon stumble upon $n=2$, which works: $7^2+24^2=49+576=625=25^2$.

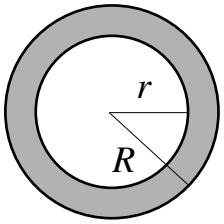
Category 2
Geometry
Meet #4, March 2001

You may use a calculator today!

1. In the square shown below, the radius of each of the small circles is 2 cm. Find the area of the shaded region. Use 3.14 as an approximate value of pi and express your answer as a decimal to the nearest hundredth of a square centimeter. The larger shape is half a circle and touches three edges and the smaller circles. The small circles touch each other and the edges.



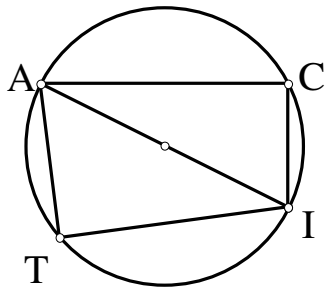
2. The two concentric circles shown below were constructed so that the area of the shaded ring equals the area of the inner circle. If the radius of the smaller circle, *r*, is 1 centimeter, what is the radius of the larger circle, *R*? Give your answer to the nearest thousandth of a centimeter.



3. In the figure below, $\overline{AI}$ is a diameter of the circle and both C and T are points on the circle. If the measure of angle CAT is 83 degrees and the measure of angle TAI is 56 degrees, find the measure of angle CIA. Express your answer in degrees.

Answers

1. _____
2. _____
3. _____



Solutions to Category 2
Geometry
Meet #4, March 2001

Answers

1. 13.76

2. 1.414

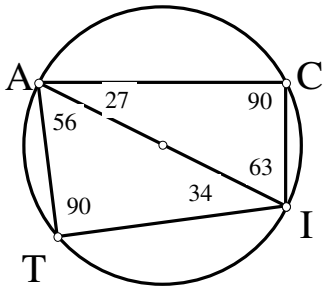
3. 63

1. The area of the entire square is $8 \times 8 = 64$ square centimeters. From this, we must subtract the area of the semicircle and each of the small circles. The area of the semicircle is $\frac{1}{2} \times 3.14 \times 4^2$ $1.57 \times 16 = 25.12$ square centimeters. The area of the two small circles is $2 \times 3.14 \times 2^2 = 6.28 \times 4 = 25.12$ square centimeters. The area of the shaded portion of the figure is thus: $64 - 2 \times 25.12 = 64 - 50.24 = 13.76$ square centimeters.

Feb 2008 note: The original problem did not say: the small circles go edge to edge; the larger semicircle is indeed half a circle; the semicircle touches three edges and touches the smaller circles, and the figure is a square. Thanks to Sean Strelow for pointing this out.

2. Since the shaded ring has the same area as the inner circle, the circle of radius *R* must have exactly *twice* the area of the inner circle. This gives us the equation: $\pi R^2 = 2\pi r^2$. Substituting $r = 1$ and solving for *R*, we get $R^2 = 2$ or $R = \sqrt{2} \approx 1.414$ to the nearest thousandth of a centimeter.

3. Since $\overline{AI}$ is a diameter of the circle and both C and T are points on the circle, the angles ACI and ATI are right angles. Angles CAT and TAI are given. The remaining angles can be worked out easily. The measure of angle CIA is 63 degrees.



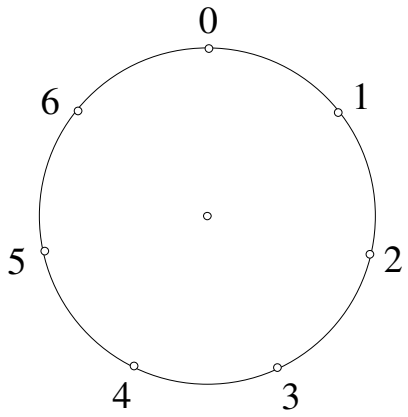
Category 3
Number Theory
Meet #4, March 2001

You may use a calculator today!

1. If 17 is the first term of the sequence shown below, find the value of the 8^{th} term.

17, 20, 26, 35, 47, …

2. On the seven-hour clock shown here, $2 \times 4 = 1$. Find the whole number value of x such that $0 \le x < 7$ and the equation $3x = 1$ is true on the seven-hour clock.



3. Jeff received a check from his grandmother for his birthday and decided to spend it all on CD's. He found that if he bought as many \$18 CD's as he could, he would have no money left over. Alternatively, if he bought as many \$15 CD's as he could, he would have \$12 left over, and if he bought as many \$13 CD's as he could, he would have \$7 dollars left over. What is the least number of dollars that Jeff could have received in the check from his grandmother?

Answers

1. _____
2. _____
3. \$ _____

Solutions to Category 3
Number Theory
Meet #4, March 2001

Answers

1. 101

2. 5

3. $72

1. The sequence can be continued to the eighth term as follows: 17, 20, 26, 35, 47, 62, 80, 101.

2. One way to solve this problem is to jump around the clock by 3's until you land on 1. That will happen on the fifth jump, so $x = 5$. Another way to solve $3x = 1$ on a seven-hour clock is to look at numbers that are *one more than a multiple of seven* until you find one that is also a multiple of three.

$1 \times 7 + 1 = 8$, but 8 is not a multiple of 3.
$2 \times 7 + 1 = 15$, and $15 = 3 \times 5$, so $x = 5$.

3. The amount of money that Jeff received from his grandmother is a multiple of 18 that is also 12 more than a multiple of 15 and 7 more than a multiple of 13. We want the least positive number that satisfies these three conditions. The following list of multiples is helpful:

Multiples of 18:	18	36	54	**72**	90
Multiples of 15:	15	30	45	**60**	75
Multiples of 13:	13	26	39	52	**65**

72 is 12 more than 60 and 7 more than 65. Jeff must have received $72 from his grandmother.

Category 4
Arithmetic
Meet #4, March 2001

You may use a calculator today!

1. A sweater that was marked $59.99 is to be sold at 30% off this original price. What is the sale price? Round your answer to the nearest penny.

2. On each of the last two math tests that Katherine has taken she got three questions wrong. One of those tests had only 12 questions, while the other had 37 questions. How many percentage points higher was Katherine's score on the test with 37 questions than it was on the test with only 12 questions? Round your answer to the nearest percent.

3. Melinda plans to invest $10,000 in a savings account that will pay one half of one percent every month on the total amount of money in the account at the end of each month. If Melinda does not make any deposits or withdrawals, what will her balance be at the end of one year? Express your answer to the nearest cent.

Answers

1. $ _____
2. _____ %
3. $ _____

Solutions to Category 4
Arithmetic
Meet #4, March 2001

Answers

1. $41.99

2. 17

3. $10,616.78

1. The sweater is marked 30% off, which means that the sale price will be 70% of the original price. $0.70 \times 59.99 = 41.993$, which is $41.99 to the nearest hundredth of a dollar.

2. We must calculate the percent values of the fractions $\frac{9}{12}$ and $\frac{34}{37}$, and find the difference. $\frac{9}{12} = 0.75 = 75\%$ and $\frac{34}{37} = 0.\overline{918} \approx 92\%$. Katherine scored 17 percentage points higher on the test with 37 questions than she did on the test with only 12 questions.

3. The balance at the end of the first month is the original deposit ($10,000) plus the interest ($\$10{,}000 \times 0.005 = \$50$), for $10,050. This can be calculated directly if we multiply the deposit by the sum *one plus the monthly interest rate* ($\$10{,}000 \times 1.005 = \$10{,}050$). Similarly, we can multiply the first month's balance by *one plus the monthly interest rate* to calculate the balance at the end of the second month ($\$10{,}050 \times 1.005 = \$10{,}100.25$). Repeating this multiplication ten more times gives a balance of $10,616.78 and the end of twelve months. The shortcut on a calculator is to multiply the original $10,000 deposit by the sum *one plus the monthly interest rate* raised to the twelfth power.

$$\$10{,}000 \times 1.005^{12} = \$10{,}616.78$$

Category 5
Algebra
Meet #4, March 2001

You may use a calculator today!

1. Sally pays a base fee of $12.95 per month for local telephone service. This plan allows her 60 minutes of local phone calls with no extra charge. If she uses more than 60 minutes on local calls, she has to pay $0.05 per minute for those additional minutes. How much will her local phone bill be if she talks for a total of 97 minutes on local calls?

2. A science teacher decided to put tennis balls on the legs of all the chairs and stools in the lab to protect the floor and to decrease noise. There are 40 seats in all and she needed 144 tennis balls to cover all the legs. If the stools have three legs and the chairs have four legs, how many stools are there in the science lab?

3. Tom must travel 90 miles and hopes to average 60 miles per hour on his trip. Unfortunately, he gets stuck in traffic early on and only averages 30 mph for the first 36 miles. What speed must Tom average for the remainder of the trip so that he still averages 60 mph for the entire trip? Express your answer in miles per hour to the nearest whole number.

Answers

1. $ _____
2. _____
3. _____ mph

Solutions to Category 5
Algebra
Meet #4, March 2001

Answers

1. \$14.80

2. 16

3. 180 mph

1. Sally will have to pay the base fee of \$12.95 plus \$0.05 per minute for each of the 37 minutes beyond the first 60 minutes. Her local phone bill will be: $\$12.95 + 37 \times \$0.05 = \$12.95 + \$1.85$ =\$14.80.

2. We have two equations and two unknowns. Let s be the number of three-legged stools and c be the number of four-legged chairs. We are told that there are 40 seats, so we have the equation $s + c = 40$. We are also told that the teacher needed 144 tennis balls to cover all the legs, so we have the equation $3s + 4c = 144$. Substituting $(40 - c)$ for the value of s in the other equation, we get:
$3(40 - c) + 4c = 120 - 3c + 4c = 120 + c = 144$,
so $c = 24$ and $s = 40 - 24 = 16$. There are 16 stools.

3. If Tom is to travel 90 miles averaging 60 miles per hour, he must complete his trip in $90 \div 60 = 1.5$ hours. Since he averages only 30 mph for the first 36 miles, we can calculate that he has already used up $36 \div 30 = 1.2$ hours. That leaves just $1.5 - 1.2 = 0.3$ hours or 18 minutes to travel the remaining $90 - 36 = 54$ miles. Tom will have to travel at a speed of $54 \div 0.3 = 180$ mph to achieve an overall average of 60 mph for the entire trip. Perhaps he should call and say that he is running late.

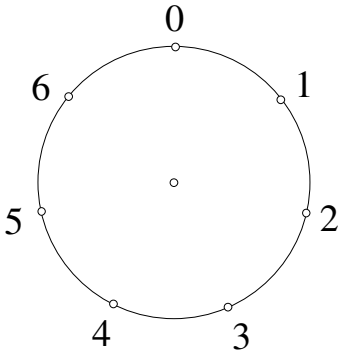
Category 6
Team Questions
Meet #4, March 2001

You may use a calculator today!

1. A four-digit number is a multiple of seven and ends in two zeros. The sum of its digits is eleven. How many factors of 2 does this number have?

2. As soon as Molly arrived on Mars, she exchanged some American dollars for Martian dollars. For every American dollar she exchanged, she received $1.25 of the Martian currency. Molly then went shopping at the Martian Market and bought a Marimba for 100 Martian dollars. How many American dollars did Molly actually spend on the Martian Marimba?

3. Find the value of 3^5 on the seven-hour clock shown at right.



4. Find the value of the following sum in base ten.

$212_{(three)} + 212_{(four)} + 212_{(five)} + 212_{(six)}$
$+ 212_{(seven)} + 212_{(eight)} + 212_{(nine)} =$ _____$_{(ten)}$

5. Earl delivered a cord of firewood that Kathy stacked neatly in two piles. One pile was 110 inches long, 40 inches high, and 16 inches deep. The other pile was 86 inches long, 54 inches high, and 16 inches deep. When Kathy finished stacking the wood, she made some calculations and found that Earl did not deliver a full cord of wood. If a cord is supposed to be 128 cubic feet, calculate, to the nearest cubic foot, how much wood Earl owes Kathy?

Answers

1. _____ = A
2. _____ = B
3. _____ = C
4. _____ = D
5. _____ = E
6. _____

6. Evaluate the following expression, using the answers to questions 1 through 5 as the values of *A, B, C, D,* and *E*, respectively:

$$\sqrt[A]{\frac{B-A}{C} - \frac{D}{E}}$$

Solutions to Category 6
Team Questions
Meet #4, March 2001

Answers

1. 5

2. $80

3. 5

4. 616

5. 44

6. 1

1. A four-digit number ending in two zeros can be thought of as a two digit number multiplied by 100. The prime factorization of 100 is $2^2 \times 5^2$, which gives us two factors of 2 already. We are now looking for a two-digit number that is a multiple of seven and whose digits add up to 11. 56 is the only two digit number that satisfies both requirements. The prime factorization of 56 is $2^3 \times 7^1$, which gives us three more factors of 2. The four-digit number 5600 has five factors of 2.

2. Converting the exchange rate 1.25 to the fraction $\frac{5}{4}$, we can see that Molly received 5 Martian dollars for every 4 American dollars she exchanged. The 100 Martian dollars that Molly spent on the Martian can be thought of as 20 groups of 5 Martian dollars which is the same as 20 groups of 4 American dollars, or $80 American. Alternatively, Molly can simply multiply any Martian price tag by the reciprocal of the exchange rate—$\left(\frac{4}{5}\right)$ in this case—to find the value in American dollars.

$$\frac{4}{5} \times \$100 \text{ Martian} = \$80 \text{ American.}$$

3. The value of 3^5 on the seven-hour clock is equal to the remainder when 3^5 is divided by seven. $3^5 = 243$ and $243 = 7 \times 34 + 5$, so $243 \div 7$ will leave a remainder of 5.

4. The first step for this problem is to convert each term of the sum to base ten.

$$212_{(three)} = (2\times 9 + 1\times 3 + 2)_{(ten)} = 23_{(ten)}$$

$$212_{(four)} = (2\times 16 + 1\times 4 + 2)_{(ten)} = 38_{(ten)}$$

$$212_{(five)} = (2\times 25 + 1\times 5 + 2)_{(ten)} = 57_{(ten)}$$

$$212_{(six)} = (2\times 36 + 1\times 6 + 2)_{(ten)} = 80_{(ten)}$$

$$212_{(seven)} = (2\times 49 + 1\times 7 + 2)_{(ten)} = 107_{(ten)}$$

$$212_{(eight)} = (2\times 64 + 1\times 8 + 2)_{(ten)} = 138_{(ten)}$$

$$212_{(nine)} = (2\times 81 + 1\times 9 + 2)_{(ten)} = 173_{(ten)}$$

Now we add up all the base-ten values of the terms, obtaining the final sum:

$$23 + 38 + 57 + 80 + 107 + 138 + 173 = 616_{(ten)}$$

5. We can calculate the volume of each of the two wood piles using the formula $V = lwh$. The first pile measures 110 inches by 40 inches by 16 inches, which is 70,400 cubic inches. The second pile measures 86 inches by 54 inches by 16 inches, which is 74,304 cubic inches. Together the two piles amount to 144,704 cubic inches of wood. Every cubic foot measures 12 inches by 12 inches by 12 inches. This means that there are 12^3 or 1728 cubic inches in every cubic foot. Dividing 144,704 cubic inches by 1728 and rounding to the nearest cubic foot, we find that Earl delivered only about 84 cubic feet of wood. Earl owes Kathy $128 - 84 = 44$ cubic feet of wood.

6. Substituting for *A* through *E* gives:

$$\sqrt[A]{\frac{B-A}{C} - \frac{D}{E}} = \sqrt[5]{\frac{80-5}{5} - \frac{616}{44}}$$

$$= \sqrt[5]{15-14} = \sqrt[5]{1} = 1$$