Intermediate Mathematics League of Eastern Massachusetts

Meet # 3
January 2001

Category 1
Mystery
Meet #3, January 2001

1. If the pattern shown in the figures below is continued, how many black squares will be needed to make the figure that uses 44 white squares?

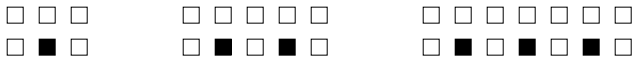


Figure 1 Figure 2 Figure 3

2. A circus clown rides his unicycle on a tight rope that is 80 feet long. He pedals forward 5 feet, then backward 2 feet to catch his balance, then forward 5 feet, then backward 2 feet, all the way across the tight rope until he reaches the platform. If you add up the distance he travels forward and the distance he travels backward, how many feet will he actually travel before he finally reaches the platform at the other end of the tight rope?

3. What is the remainder when you divide the 8^{th} power of 2 by the 3^{rd} power of 5?

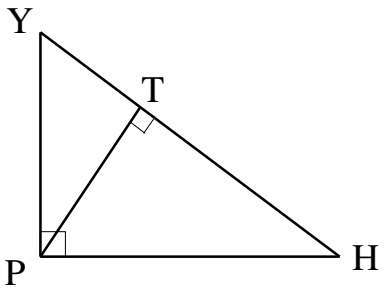
Answers

1. _____
2. _____
3. _____

Category 2
Geometry
Meet #3, January 2001

1. How many inches are in the side length of a square that has half the area of the parallelogram shown below?

2. Triangle PYTH is a right triangle with $\overline{PY} = 6$ cm and $\overline{PH} = 8$ cm. How many centimeters long is $\overline{PT}$, which is an altitude relative to side $\overline{YH}$?



3. Mr. Snood packed his homing pigeon, Homer, in the car one Saturday morning and set out for a drive. He drove 7 miles due west, then 6 miles due south, then 14 miles due west, then 4 miles due south, and finally 3 more miles due west. He then released Homer from his cage and the bird flew off toward home. How many miles will Homer have to fly if he flies directly back to the house?

Answers

1. _____
2. _____
3. _____

Category 3
Number Theory
Meet #3, January 2001

1. Express the base 4 numeral 3132 as a base 8 numeral.

2. In a certain base number system (base B), $5+5+5+5+5+5+5=55_B$.
If 6×5 is 30 in base ten, how would we write this value in base B?

3. Simplify the expression below. Write your result in scientific notation.

$$\frac{\left(9.6\times10^{29}\right)\cdot\left(1.32\times10^{-11}\right)}{\left(1.2\times10^{23}\right)\cdot\left(1.6\times10^{-8}\right)}$$

Answers

1. _____
2. _____base B
3. _____

Category 4
Arithmetic
Meet #3, January 2001

1. Evaluate the expression. Write your result in simplest form.

$$\left(2^2\right)\cdot\left(3^{-4}\right)\cdot\left(5^2\right)\cdot\left(9^2\right)\cdot\left(2^{-3}\right)\cdot\left(7^2-1\right)\cdot\left(10^{-2}\right)$$

2. How many whole numbers are between $\sqrt[4]{100}$ and $\sqrt[2]{1000}$?

3. Evaluate the following expression. Write your answer in simplest radical form.

$$\sqrt[6]{\left(\sqrt{81}\times\sqrt{64}\right)^3}+\sqrt[2]{\left(2\times\sqrt[3]{512}+\sqrt{256}\right)}$$

Answers
1. _____
2. _____
3. _____

Category 5
Algebra
Meet #3, January 2001

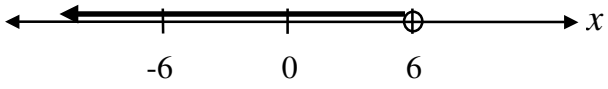
1. Find the positive difference between the two values of x that make the following equation true.

$$|2x - 7| = 19$$

2. Find the set of all the integer values of x, such that

$$\left|\frac{10}{x}\right| > 3$$

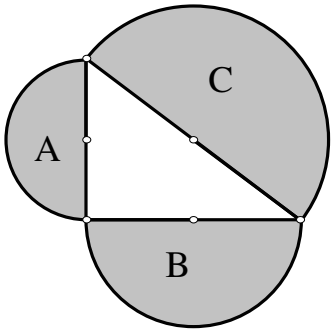
3. The solution set of the equation $9x + 3(6 - x) - C < 10$ is given by the graph shown below. What value of C will make this true?



Answers

1. _____
2. $x = \{$ _____ $\}$
3. _____

Category 6
Team Questions
Meet #3, January 2001



1. Semicircles A, B, and C are constructed on the sides of a right triangle. If the area of semicircle A is 18π square units and the area of semicircle C is 50π square units, how many square units are in the area of semicircle B? Express your answer in terms of pi in simplest form.

2. If AB is a 2-digit number where A is the tens digit and B is the ones digit, what is the greatest possible value of $B^2 - A^2$?

3. What is the units digit of the 59^{th} power of 7?

4. Both $x = 2$ and $x = {}^{-}2$ are solutions to one of the equations below, but $x = 8$ is *not* a solution to that same equation. Which of the following could be the equation? Write the number of the correct equation.

1. $|x-6| \geq 8$ 2. $|x-6| \leq 8$ 3. $|x-8| \geq 6$ 4. $|x-8| \leq 6$
5. $|x-6| > 8$ 6. $|x-6| < 8$ 7. $|x-8| > 6$ 8. $|x-8| < 6$

5. There are roughly 9.8×10^{23} molecules of H_2O in one fluid ounce of water. Using this approximate value, calculate how many molecules of water there are in one gallon? Express your answer in scientific notation without any further rounding. (Reminder: There are 8 fluid ounces in a cup, 2 cups in a pint, 2 pints in a quart, and 4 quarts in a gallon.)

Answers

1. _____ = A
2. _____ = B
3. _____ = C
4. _____ = D
5. _____ ≠ E
6. _____

6. Evaluate the following expression, using the answers to questions 1 through 4 as the values of *A, B, C,* and *D*, respectively. For the value of *E*, use the ***exponent*** of the answer to question 5.

$$\sqrt[D]{\left(\frac{B+1}{CD}\right) + \sqrt{\left(\frac{10A}{B\pi}\right)^{C}} + \left(\frac{E}{2} - D\right)}$$

Solutions to Category 1
Mystery
Meet #3, January 2001

Answers

1. 14

2. 180

3. 6

1. Each new figure in the pattern has one more black and three more white squares than the previous figure. Since figure 1 started with one black square and five whites, we could say that figure 0 would have just two white squares. This reasoning helps us to write the formula $w = 2 + 3b$, where w is the number of white squares and b is the number of black squares. (Note that the number of black squares is the same as the figure number.) Now we can solve for b when $w = 44$.

$$44 = 2 + 3b$$
$$42 = 3b$$
$$b = 14$$

The figure that uses 44 white squares will require 14 black squares.

2. Each time the clown pedals forward 5 feet and back 2 feet, he travels a total of 7 feet, but he only makes 3 feet of progress across the tightrope. He will repeat this pattern 25 times until he is 75 feet across the tightrope, then he will reach the other platform in a final forward motion, without rolling back. Thus the total distance traveled is: $7 \times 25 + 5 = 180$ feet. Alternatively, we can say that he will travel the full 80 feet of the rope plus two times the amount that he goes backward. Each time he goes backward he will have to cover that same distance again going forward. This reasoning gives us $80 + 4 \times 25 = 180$ feet.

3. The 8^{th} power of 2 is 256 and the 3^{rd} power of 5 is 125. Thus 256 divided by 125 leaves a remainder of 6.

Solutions to Category 2
Geometry
Meet #3, January 2001

Answers

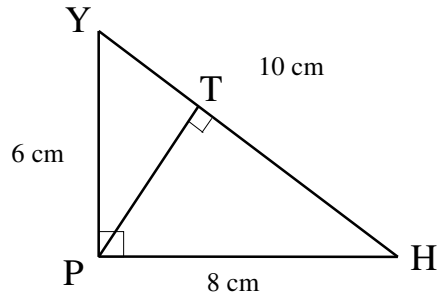
1. 7

2. 4.8

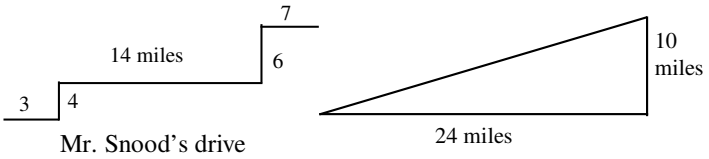
3. 26

1. The area of the parallelogram is $12.25 \times 8 = 98$ square inches. Half of 98 is 49, so the square has a side length of 7 inches.

2. Using the Pythagorean theorem, we can find the length of $\overline{YH}$. $6^2 + 8^2 = 36 + 64 = 100$, so $\overline{YH} = \sqrt{100} = 10$ cm. The area of the triangle is $\frac{1}{2} \times 6 \times 8 = 24$ square centimeters. Using $\overline{YH}$ as base and the unknown $\overline{PT}$ as height, we have $\frac{1}{2} \times 10 \times \overline{PT} = 24$. $\overline{PT} = 4.8$ cm.



3. We can swap Mr. Snood's actual drive for the right triangle shown below. He drove a total of 24 miles due west and 10 miles due south, so Homer's flight is the hypotenuse of the right triangle.



Using the Pythagorean Theorem, we calculate:
$x^2 = 10^2 + 24^2 = 100 + 576 = 676,\ x = \sqrt{676} = 26$

Solutions to Category 3
Number Theory
Meet #3, January 2001

Answers

1. 336

2. 50

3. 6.6×10^3

1. The base 4 numeral 3132 can be thought of as the sum of its four parts, each digit times its place value: $(3\times64)+(1\times16)+(3\times4)+(2\times1)$. Our number in base 8 will be a three digit number, abc, where $(a\times64)+(b\times8)+(c\times1)$ is the base-10 value of the number. We can put the (3×64) directly into base 8 as a 3 in the third place from the left. Rearranging the (1×16) as (2×8) and the (3×4) as $(1\times8)+(4\times1)$, we have 3 eights for the second place from the left. For the units place we have (4×1) and (2×1), which puts a 6 in the units place. Our base 8 number is 336.

2. It is not necessary to find out what base *B* is to solve this problem. We are told that seven 5's is 55_B. If we subtract one of those 5's, we will have the sum of six 5's, which is the same as 6×5. It is easy to subtract 5 from 55_B since 5 is the digit in the units place and we don't have to "borrow." Thus, $55_B-5=50_B$. Some students may go to the trouble of finding out what base *B* is anyway. Since $5\times7=35$ in base 10, we know that $55_B=35_{10}$. This means that $5\times B+5=3\times10+5$, or $5B=30$, and base *B* must be base 6.

3. $$\frac{\left(9.6\times10^{29}\right)\cdot\left(1.32\times10^{-11}\right)}{\left(1.2\times10^{23}\right)\cdot\left(1.6\times10^{-8}\right)}=\frac{9.6}{1.2}\cdot\frac{1.32}{1.6}\cdot\frac{10^{(29-11)}}{10^{(23-8)}}$$

$$=\frac{6\times1.6}{1.2}\cdot\frac{1.2\times1.1}{1.6}\cdot\frac{10^{18}}{10^{15}}=6\times1.1\times10^3=6.6\times10^3$$

Solutions to Category 4
Arithmetic
Meet #3, January 2001

Answers

1. 6

2. 28

3. $10\sqrt{2}$

1. A convenient first step for this problem is to gather together powers with the same base.

$$\left(2^2\right)\cdot\left(3^{-4}\right)\cdot\left(5^2\right)\cdot\left(9^2\right)\cdot\left(2^{-3}\right)\cdot\left(7^2-1\right)\cdot\left(10^{-2}\right)$$
$$=\left(2^2\right)\cdot\left(2^{-3}\right)\cdot\left(3^{-4}\right)\cdot\left(9^2\right)\cdot\left(5^2\right)\cdot\left(10^{-2}\right)\cdot\left(7^2-1\right)$$

Looking at just the first two parts we have:

$$\left(2^2\right)\cdot\left(2^{-3}\right)=\left(2^{2-3}\right)=\left(2^{-1}\right)$$

The third and fourth parts clean up nicely:

$$\left(3^{-4}\right)\cdot\left(9^2\right)=\left(3^{-4}\right)\cdot\left(3^4\right)=\left(3^{4-4}\right)=\left(3^0\right)=1$$

The fifth and sixth parts become:

$$\left(5^2\right)\cdot\left(10^{-2}\right)=\left(5^2\right)\cdot\left((2\cdot5)^{-2}\right)=\left(5^{2-2}\right)\cdot\left(2^{-2}\right)=\left(2^{-2}\right)$$

In the end, we have:

$$\left(2^{-3}\right)\cdot\left(7^2-1\right)=\frac{48}{8}=6$$

2. Whatever the value of $\sqrt[4]{100}$ is, it must be between 3 and 4, since $3^4=81$ and $4^4=256$. Similarly, $\sqrt[2]{\sqrt[3]{1000}}$ must be between 31 and 32, since $31^2=961$ and $32^2=1024$. Thus the whole numbers between $\sqrt[4]{100}$ and $\sqrt[2]{\sqrt[3]{1000}}$ begin with 4 and end with 31. That's 28 numbers.

3. Fortunately, the numbers involved in this expression are all powers of 2 and 3.

$$\sqrt[6]{\left(\sqrt{81}\times\sqrt{64}\right)^3}+\sqrt[2]{\left(2\times\sqrt[3]{512}+\sqrt{256}\right)}$$

$$\sqrt[6]{(9\times8)^3}+\sqrt[2]{(2\times8+16)}=\sqrt[2]{72}+\sqrt[2]{32}$$

$$=6\sqrt{2}+4\sqrt{2}=10\sqrt{2}$$

Solutions to Category 5
Algebra
Meet #3, January 2001

Answers

1. 19

2. $x = \{^{-}3, ^{-}2, ^{-}1, 1, 2, 3\}$

3. 44

1. To solve an equation with absolute value we must pursue two possibilities. In this case, $|2x - 7|$ could equal 19 or $^{-}19$.

$2x - 7 = 19$ $\qquad$ $2x - 7 = ^{-}19$

$2x = 26$ $\qquad$ $2x = ^{-}12$

$x = 13$ $\qquad$ $x = ^{-}6$

The positive difference between these two solutions is 19.

2. The absolute value of x must remain small so that the quotient remains greater than 3. The six possibilities are as follows:

$x = ^{-}3, \left|\frac{10}{^{-}3}\right| = 3\frac{1}{3} > 3$ $\qquad$ $x = ^{-}2, \left|\frac{10}{^{-}2}\right| = 5 > 3$

$x = ^{-}1, \left|\frac{10}{^{-}1}\right| = 10 > 3$ $\qquad$ $x = 1, \left|\frac{10}{1}\right| = 10 > 3$

$x = 2, \left|\frac{10}{2}\right| = 5 > 3$ $\qquad$ $x = 3, \left|\frac{10}{3}\right| = 3\frac{1}{3} > 3$

The set, then, is $x = \{^{-}3, ^{-}2, ^{-}1, 1, 2, 3\}$.

3. First we distribute and combine like terms.

$$9x + 3(6 - x) - C < 10$$

$$9x + 18 - 3x - C < 10$$

$$6x < C - 8$$

$$x < \frac{C - 8}{6}$$

We see from the line graph that the solution is $x < 6$. That means that $\frac{C - 8}{6} = 6$. Now we solve for C. $C - 8 = 36$, so $C = 44$.

Solutions to Category 6
Team Questions
Meet #3, January 2001

Answers

1. 32π

2. 80

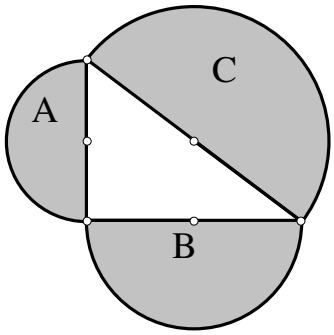
3. 3

4. 3

5. 1.2544×10^{26}

6. 3

1. The Pythagorean Theorem tells us that the sum of the squares on the legs equals the square on the hypotenuse of a right triangle. Well, this is true of semicircles too. Students who know this can simply calculate $50\pi - 18\pi = 32\pi$, and they'll have the correct answer. Other students will go to the trouble of finding the lengths of the three sides and finding the area of semicircle B from there.



Semicircle A with an area of 18π square units is half of a circle of 36π square units. The radius of that circle would be 6 units and the diameter 12 units. Similarly, semicircle C with an area of 50π square units comes from a full circle of area 100π square units, with a radius of 10 units, and a diameter of 20 units. We now have two of the three sides of the triangle. To obtain the third side we use the Pythagorean Theorem.

$$12^2 + x^2 = 20^2$$
$$144 + x^2 = 400$$
$$x^2 = 256$$
$$x = 16$$

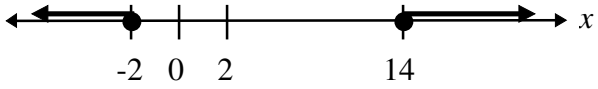
This 16 units is the diameter of semicircle B, so the radius is 8, and the area is half of 64π, or 32π.

2. We want A^2 to be small and B^2 to be big, so the two digit number 19 will give the largest difference. $9^2 - 1^2 = 81 - 1 = 80$

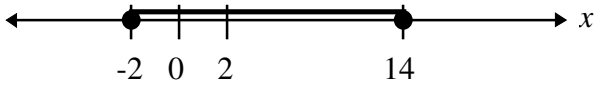
3. The first four powers of 7 are 7, 49, 343, 2401. Since 2401 ends in 1, the next power of 7 will end in 7 and the cycle will repeat with a period of four. The units digit of the 59^{th} power of seven will be the third digit in this cycle, since $59 = 4 \times 14 + 3$. The third digit in the cycle is 3.

4. To answer this question, it would help to graph the equations. It is not necessary to graph all eight equations, since the bottom four are the same as the top four except that the end points are not included. Here are the line graphs of equations 1 through 4.

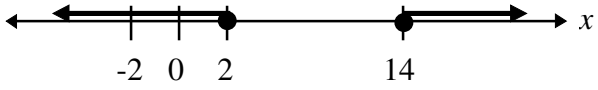
1. $|x - 6| \geq 8$



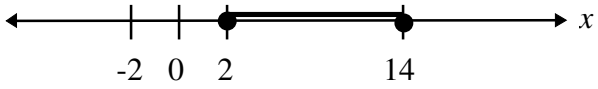
2. $|x - 6| \leq 8$



3. $|x - 8| \geq 6$



4. $|x - 8| \leq 6$



Graph C shows $x = 2$ and $x = {}^{-}2$ are solutions, but $x = 8$ is not. The graph for equation G would have open circles at 2 and 14, so $x = 2$ would not be a solution to equation G.

5. We must multiply 9.8×10^{23} by 8 and then by 2, 2, and 4. This can be done in any order, but a simple way is to multiply 9.8 by the 128 ounces in a gallon and then consider what power of 10 is required. $9.8\times128=1254.4$. If we multiply this value by 10^{23}, we will have the correct value, but the number will not be in scientific notation since 1254.4 is larger than 10. Dividing 1254.4 by 1000 and raising the exponent by three, we get the proper scientific form which is 1.2544×10^{26}**.**

6. Substitute the correct values for *A* through *E*, then simplify:

$$\sqrt[D]{\left(\frac{B+1}{CD}\right)+\sqrt{\left(\frac{10A}{B\pi}\right)^{C}}+\left(\frac{E}{2}-D\right)}$$

$$=\sqrt[3]{\left(\frac{80+1}{3\cdot3}\right)+\sqrt{\left(\frac{10\cdot32\pi}{80\pi}\right)^{3}}+\left(\frac{26}{2}-3\right)}$$

$$=\sqrt[3]{\left(\frac{81}{9}\right)+\sqrt{(4)^{3}}+(13-3)}$$

$$=\sqrt[3]{9+8+10}=\sqrt[3]{27}=3$$