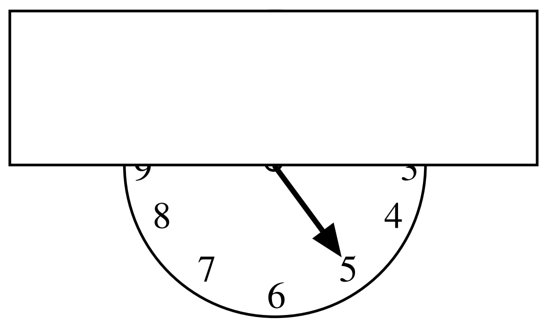
Meet # 1
October, 2000

Intermediate
Mathematics League
of
Eastern Massachusetts

Meet # 1
October, 2000

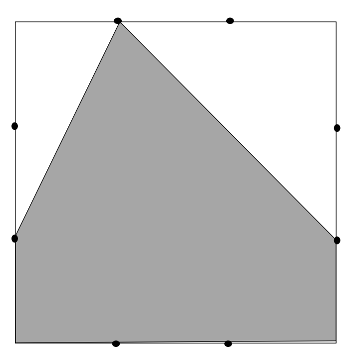
Category 1
Mystery
Meet #1, October, 2000

1. In the picture shown below, the top half of the clock is obstructed from view and the bottom half shows only that the minute hand is on the five. If it is after sunrise and before sunset, what is the positive difference between the earliest time and the latest time it could be?



2. Three different positive whole numbers have an average of 16. The greatest of them is one less than a perfect cube. The least of them is 5 less than half the greatest. What are the three numbers?

3. Each side of a square is trisected at the points shown. What part of the square is shaded? Express your answer as a fraction is simplest terms.



Answers

1. ______________
2. ______________
3. ______________

Solutions to Category 1
Mystery
Meet #1, October, 2000

Answers

1. 5

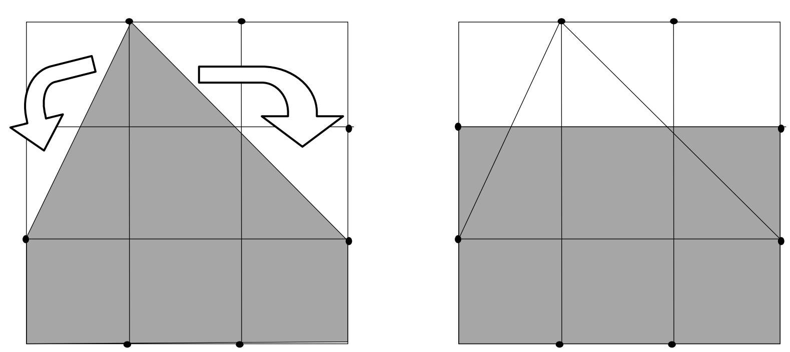
2. 8, 14, 26

3. $\frac{2}{3}$

1. The earliest it could be is 9:25 AM and the latest is 2:25 PM, which is a positive difference of 5 hours. The hour hand would be visible at 25 past any other hour.

2. Three numbers whose average is 16 must have a sum of 3×16 or 48, The first three perfect cubes are 1, 8, and 27. Since we're looking for the largest of the three numbers, it must be one less than 27, which is 26. Five less that half of 26 is 8, which gives the least of the three numbers. The remaining number must be $48 - (26 + 8) = 14$. Thus, the three numbers are 8, 14, and 26.

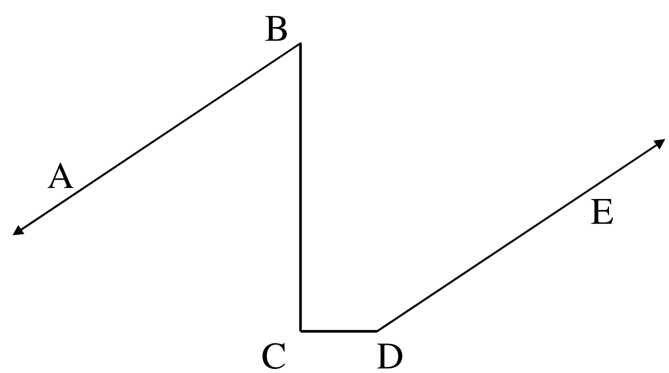
3. It may help to connect the points to make a three-by-three grid on the square. There are clearly nine smaller square units. The bottom row shows a solid three units. The triangle above the bottom row has a base of 3 and a height of 2, for another 3 square units. This is a total of 6 of the nine small squares, so $\frac{6}{9}$ or $\frac{2}{3}$ of the large square is shaded. One can also imagine flipping down the two little triangles on the top row, completing the middle row, as suggested by the figure on the right below.



Category 2
Geometry
Meet #1, October, 2000

1. A regular octagon and a regular hexagon share a common side. What is the number of degrees in the measure of the exterior angle formed where they meet?

2. In the figure, rays $\overrightarrow{AB}$ and $\overrightarrow{DE}$ are parallel. Angle C is a right angle and angle B measures 52 degrees. Find the measure of angle D if it is less than 180 degrees.



3. How many degrees are in an exterior angle of a regular 18-gon?

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 2
Geometry
Meet #1, October, 2000

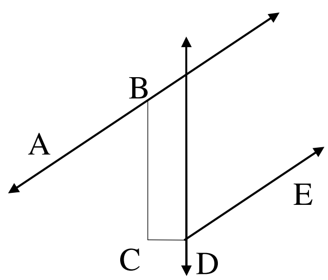
Answers

1. 105 degrees

2. 142 degrees

3. 20 degrees

1. The interior angle measures of the regular octagon and the regular hexagon are 135 degrees and 120 degrees respectively. $135 + 120 = 255$, so the exterior angle between the shapes must account for the remaining 105 degrees of a full circle.



2. By extending ray AB and creating a line parallel to segment BC, we can see that angle D is composed of a 90 degree angle and the same 52 degrees that is found at B. Thus, angle D is 142 degrees.°

3. A regular 18-gon can be divided into 16 triangles each with an angle sum of 180 degrees. The total angle sum of the 18-gon is $16 \times 180 = 2880$. Since the 18-gon is regular, each interior angle is one eighteenth of 2880, or 160 degrees. An exterior angle is the supplement of the interior angle, which is 20 degrees in this case.

Alternatively, some students will know that the sum of all the exterior angles of a polygon is always 360. Since the 18-gon is regular, we simply divide as follows: $360 \div 18 = 20$.

Category 3
Number Theory
Meet #1, October, 2000

1. For how many positive integral values of n will $\frac{168}{n}$ be a whole number?

2. What is the greatest integer that will always divide the product of four consecutive integers?

3. Find the sum of all the positive integers less than 1000 that are both perfect squares and perfect cubes.

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 3
Number Theory
Meet #1, October, 2000

Answers

1. 16

2. 24

3. 794

1. The positive integral values of *n* that will make $\frac{168}{n}$ a whole number are the factors of 168. (By "positive integral values" we mean positive integers or whole numbers.) We can simply list all the factors of 168 in pairs: 1×168, 2×84, 3×56, 4×42, 6×28, 7×24, 8×21, and 1×168. Those are the 16 values of *n* that will make $\frac{168}{n}$ a whole number. Alternatively, there is a trick for figuring out the number of factors of a number without actually listing them. We express 168 in prime factors with exponents as follows: $168=2^3\cdot3^1\cdot7^1$. We then raise each exponent by one and multiply them like this: $(3+1)\times(1+1)\times(1+1)=4\times2\times2=16$.

2. The product of four consecutive integers is guaranteed to contain a multiple of 3, and two multiples of 2, one of which is also a multiple of 4. Being careful not to count the same factor of 2 as both a multiple of 2 and a multiple of 4, we can be certain of only three factors of 2 along with the one factor of 3. From these factors we obtain the product $2^3\times3=24$, so 24 will always divide the product of four consecutive integers. Some people will find this answer simply by multiplying the first four counting numbers: $1\times2\times3\times4=24$.

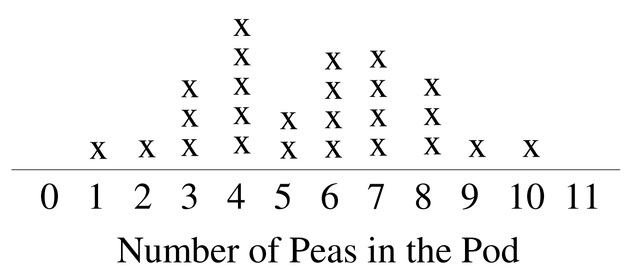
3. Numbers that are both perfect squares and perfect cubes are sixth powers, because $a^6=\left(a^2\right)^3=\left(a^3\right)^2$. Since the cube root of 1000 is 10, we need only values under 10 for a^2. Now if a^2 is under 10, then *a* must be 3 or less. The three sixth powers under 1000 are: $1^6=1$, $2^6=64$, and $3^6=729$. Their sum is $1+64+729=794$.

Category 4
Arithmetic
Meet #1, October, 2000

1. Sly knows that his quiz scores were 92, 85, 96 82, 88, and 91 for the first quarter, but he can’t remember what he earned on the test. His teacher told him that his combined average for the quizzes and the test is exactly 90 and that she counts the test as two quizzes. What score did Sly get on his test?

2. Old McDonald has 7 cats and 4 dogs. The average weight of the cats is 12 pounds and the average weight of the dogs is 33 pounds. What is the average weight of Old McDonald’s eleven animals? Round your answer to the nearest tenth of a pound.

3. The line plot shows the data Mr. Jones collected on his peas. Each x represents a pod that had the given number of peas in it.



A = the mode of the data set
B = the range of the data set
C = the median of the data set

Find the value of $\frac{2B+4C-A^2}{2A-C}$.

Answers
1. ______________
2. ______________
3. ______________

Solutions to Category 4
Arithmetic
Meet #1, October, 2000

Answers

1. 93

2. 19.6

3. 13

1. Sly's overall average of 90 comes from six quizzes and one test, which counts as two quizzes. We multiply 90 by 8 to find the sum of these eight scores, which is 720. Since the six quiz scores add up to 534, the difference $720 - 534$, or 186, must be the test score counted twice. Dividing by 2, we find that Sly must have earned a 93 on the test.

2. The total weight of the animals can be found without knowing the weights of any of the individual cats or dogs. We multiply the number of cats by their average weight and the number of dogs by their average weight and then add these together. Thus we have $7 \times 12 = 84$ pounds in cats and $4 \times 33 = 132$ pounds in dogs, which is a total of $84 + 132 = 216$ pounds of animals. Dividing 216 by 11 and rounding to the nearest tenth, we find that the average weight of Old McDonald's animals is 19.6 pounds.

3. $A = 4$ (the mode of the data set); $B = 9$ (the range of the data set); $C = 6$ (the median of the data set).

Substituting these values into $\frac{2B + 4C - A^2}{2A - C}$ gives us: $\frac{2 \cdot 9 + 4 \cdot 6 - 4^2}{2 \cdot 4 - 6} = \frac{18 + 24 - 16}{8 - 6} = \frac{26}{2} = 13$

Category 5
Algebra
Meet #1, October, 2000

1. If $A \otimes B$ means $\frac{3A^2 - B^3}{20}$, then find the value of $5 \otimes (7 \otimes 3)$. Express your answer as a mixed number in simplest form.

2. Find the value of C so that the equation below will be an identity.
(An identity is an equation in which any value of the variable will make the equation a true statement.)

$$27x - 8(3x + 5) = 4(7x + C) - 5(5x + 4)$$

3. Evaluate the expression if $x = \frac{2}{7}$.

$$5(13x - 8) + 2(13x - 8) - 4(13x - 8) + 3x + 32$$

Answers
1. _______________
2. _______________
3. _______________

Solutions to Category 5
Algebra
Meet #1, October, 2000

Answers

1. $-7\frac{1}{20}$

2. -5

3. 20

1. To find $5 \otimes (7 \otimes 3)$, we must first compute $7 \otimes 3$. Substituting 7 for *A* and 3 for *B*, we obtain

$$\frac{3 \cdot 7^2 - 3^3}{20} = \frac{3 \cdot 49 - 27}{20} = \frac{147 - 27}{20} = \frac{120}{20} = 6.$$

Now we compute $5 \otimes 6$:

$$\frac{3 \cdot 5^2 - 6^3}{20} = \frac{3 \cdot 25 - 216}{20} = \frac{75 - 216}{20} = \frac{-141}{20} = -7\frac{1}{20}$$

2. To solve this problem we first distribute (watching out for negative signs), then combine like terms:

$$27x - 8(3x + 5) = 4(7x + C) - 5(5x + 4)$$

$$= \quad 27x - 24x - 40 = 28x + 4C - 25x - 20$$

$$= \quad 3x - 40 = 3x + 4C - 20$$

Notice that we have $3x$ on both sides of the equation. Now we have to make the rest of the equation balance; we need to find *C* so that $-40 = 4C - 20$. Adding 20 to both sides and dividing by 4, we find that *C* must be -5.

3. The observant algebra student will notice that the three sets of parentheses contain the same $13x - 8$. This student would then simplify the expression before substituting. There are $5 + 2 - 4$ or 3 of those $(13x - 8)$'s.

$$5(13x - 8) + 2(13x - 8) - 4(13x - 8) + 3x + 32$$

$$= \quad 3(13x - 8) + 3x + 32$$

Then we distribute and combine like terms.

$$= \quad 39x - 24 + 3x + 32$$

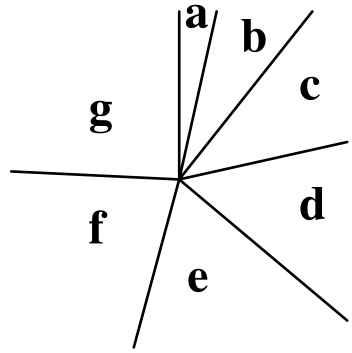
$$= \quad 42x + 8$$

And finally we substitute $\frac{2}{7}$ for *x*.

$$= \quad 42 \cdot \frac{2}{7} + 8 = 12 + 8 = 20$$

Category 6
Team Questions
Meet #1, October, 2000

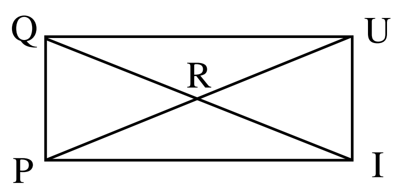
1. In the figure at right, angle a is 13º, angle b is 26º, and each successive angle is the next multiple of 13º until angle g which is simply the rest of the circle. What is g – c?



2. The difference between the squares of two consecutive odd numbers is 64. Which of the two numbers is prime?

3. How many whole numbers between 200 and 999 (excluding 200 and 999) do not contain the digits 3, 5, or 7?

4. If QUIP is a rectangle and the measure of angle QUR is 26 degrees, what is the measure of angle IRP?



5. The sum of a set of consecutive counting numbers is 60. What is the least number that could be a member of the set of numbers?

Answers
1. __________ = A
2. __________ = B
3. __________ = C
4. __________ = D
5. __________ = E
6. ____________

6. Using the values you obtained in questions 1 through 5, evaluate the following expression:

$$\frac{A+B+C-5E}{\sqrt{2D}}$$

Solutions to Category 6
Algebra
Meet #1, October, 2000

Answers

1. 48

2. 17

3. 243

4. 128

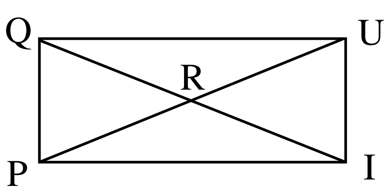
5. 4

6. 18

1. The first six multiples of 13 are: 13, 26, 39, 52, 65, and 78. Their sum is 273, so angle g is 87º. Thus g – c = 87 – 39 = 48

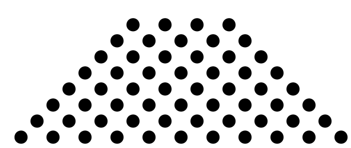
2. Trial and error would probably be the quickest way to answer this question. A student might first try $11^2 - 9^2 = 121 - 81 = 40$ and find that she needed to try larger numbers two get a larger difference. Eventually she would find that $17^2 - 15^2 = 64$ is the equation that works, and 17 is the number that is prime. Using algebra, one might assign $2n - 1$ as the smaller odd number and $2n + 1$ as the larger. Then
$(2n+1)^2 - (2n-1)^2 = 64$ or
$\left(4n^2 + 4n + 1\right) - \left(4n^2 - 4n + 1\right) = 64$ or
$8n = 64$, so $n = 8$. If *n* is 8, then $2n - 1$ is 15 and $2n + 1$ is 17. Of these two, 17 is the prime.

3. The hundreds place can have 2, 4, 6, 8 or 9, which is 5 possible digits. The tens and ones place can have 0, 1, 2, 4, 6, 8, or 9, which is 7 possible digits in each of two places. Combining these in all possible ways will yield 5 x 7 x 7 or 245 different numbers from 200 to 999 *including 200 and 999* which do not use the digits 3, 5, or 7. If we exclude 200 and 999 we have 245 – 2 = 243.



4. The measure of angle UQR is 26 degrees, just as angle QUR. Those two together amount to 52 degrees of triangle QUR, so angle QRU must be the remaining 128 degrees of the triangle. The vertical angle IRP is also 128 degrees.

5. Trial and error is probably the way to go for this problem as well. Sets of consecutive numbers beginning with 1, yield the triangular numbers, which may be familiar to some students. The first eleven triangular numbers are: 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, and 66. We see that no set of consecutive counting numbers that starts with 1 will have a sum of 60. If we start the set with 2 and add consecutive counting numbers, we get a triangular number less the 1 that was not added. The first ten of these are: 2, 5, 9, 14, 20, 27, 35, 44, 54, and 65. Again, we missed 60. One could continue in this way, starting with 3 and adding consecutive counting numbers, then starting with 4, etc. But looking more closely at that first list of triangular numbers, one might notice that 66 is exactly 6 more than the sum of 60 that we want, and 6 is itself a triangular number. In other words, we take the 1, 2, and 3 away from the sum of the first eleven counting numbers and we have a set that starts with 4 and has a sum of 60. To verify, we can check that $4+5+6+7+8+9+10+11$ is indeed 60, and the least number of this set is 4. This truncated triangular number is depicted by the dots below.



6. $$\frac{A+B+C-5E}{\sqrt{2D}}=\frac{48+17+243-5\cdot 4}{\sqrt{2\cdot 128}}$$

$$=\frac{308-20}{\sqrt{256}}=\frac{288}{16}=18$$