1997

INTERMEDIATE MATH LEAGUE OF EASTERN MASSACHUSETTS

MEET #2

Category 1 - Mystery 1997
MEET #2

1. A soda machine dispenses 90 cans of soda in 6 hours. How many cans of soda could seven soda machines dispense in 5 hours? (All machines dispense cans at the same rate.)

2. Leif works for a local landscaper. He earns \$7 per hour for the first six hours, and \$10 per hour for every hour after the first six. He earned \$79.50 today. He started work at 8:00 A.M. He took a 20-minute lunch break near noontime, for which he was not paid. At what time, P.M., did he finish work?

3. Jack lost his marbles. Some were blue, some were red, and the rest were green. 3/8 of all the marbles were blue, 3/8 of the others were red. There were 150 green marbles. How many red marbles were there?

ANSWERS

1. _______ cans
2. _______ P.M.
3. _______

SOLUTIONS - MEET #2

1997

CATEGORY 1

1. 525
2. 6:05
3. 90

(1) $90 \div 6$ is 15 cans per hour. In 5 hours, a soda machine could dispense $15 \cdot 5$, or 75 cans. Seven machines could dispense $7 \cdot 75$, or 525 cans.

(2) \$7 x 6 = \$42 for the first 6 hours.
\$79.50 − \$42 = \$37.50 for the last x hours
@ \$10/hr, so $37.50 \div 10 = 3.75$ hours.
So, Leif worked for 6 + 3.75, or 9 hr., 45 min.
Add on the 20 minutes unpaid lunch break. If he started work at 8:00 A.M., then he finished 10 hr, 5 min. later, at 6:05 P.M.

(3) 3/8 of total = blue
3/8 of rest = red = 3/8 of 5/8, or 15/64

$$\text{Green} = \frac{64}{64} - (\text{blue} + \text{red})$$
$$= \frac{64}{64} - \left(\frac{3}{8} + \frac{15}{64}\right)$$
$$= \frac{64}{64} - \left(\frac{24}{64} + \frac{15}{64}\right)$$
$$= \frac{64}{64} - \frac{39}{64}$$
$$= \frac{25}{64}$$

150 green marbles = $\frac{25}{64}$ of all the marbles,
so $150 \div 25$, or 6, = $\frac{1}{64}$ of all the marbles.
Red = $\frac{15}{64}$, so $15 \cdot 6 = 90$ red marbles.

Students with some algebra know-how may do something like this:

Let x = total # of marbles.
Red + Blue + Green = Total

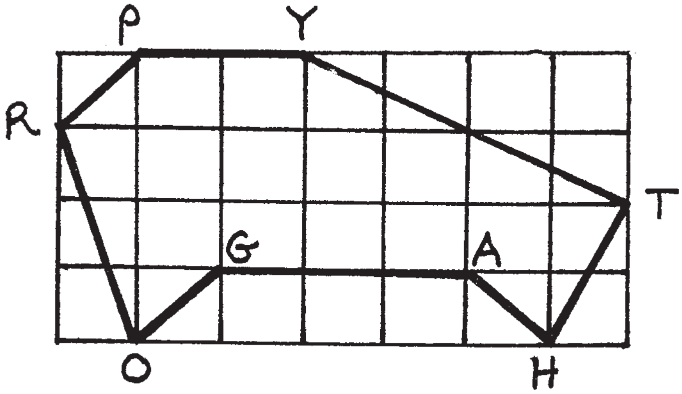
$$\frac{3}{8}\left(\frac{5}{8}x\right) + \frac{3}{8}x + 150 = x$$
$$\frac{15}{64}x + \frac{24}{64}x + 150 = x$$
$$\frac{39}{64}x + 150 = x$$

$$150 = \frac{25}{64}x$$
$$\frac{64}{\cancel{25}_{1}} \cdot \cancel{150}^{6} = \frac{64}{25} \cdot \frac{25}{64}x$$
$$384 = x$$
$$\text{Red} = \frac{15}{\cancel{64}_{1}}(\cancel{384}^{6})$$
$$= 90$$

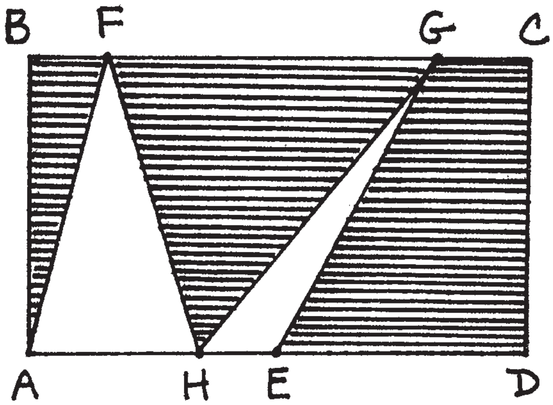
Category 2 - GEOMETRY 1997
MEET # 2

① An equilateral triangle and a square have the same perimeter. The area of the square is 64 square inches. How many inches are in the perimeter of the triangle?

② Each small square has a length of 4 centimeters. How many square centimeters are in the area of polygon PYTHAGOR?



③



ABCD is a rectangle. F and G are points on $\overline{BC}$. H is a point on $\overline{AD}$ between A and E. E is the midpoint of $\overline{AD}$.
AB = 12 inches.
BC = 30 inches.
Find the number of square inches in the shaded area.

ANSWERS

① ______ inches

② ______ sq. cm

③ ______ sq. in.

Geometry

SOLUTIONS - Meet #2 1997

CATEGORY 2

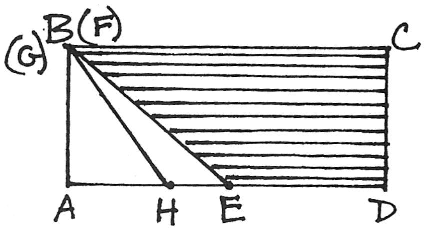
1. 32
2. ~~17~~ 272
3. 270

① A square whose area is 64 in.² has a length of $\sqrt{64}$, or 8 inches. Its perimeter is $8 \cdot 4$, or 32". The triangle has the same perimeter as the square, so its perimeter is also 32".

② The polygon's region may be subdivided into rectangles and right triangles, each of whose areas is easily computed or counted.
17 squares total, each of which is 4x4=16 cm², so 17x16=272

③ Key concept: A triangle's area is the product $\frac{1}{2}bh$, where b is its base and h is its altitude. Also, triangles of the same base and altitude have the same area.
The shaded area may be calculated by subtracting the unshaded area from the area of the rectangle.

The unshaded area may be seen more easily as follows: If F or G were at any point on $\overleftrightarrow{BC}$, or even at the same point, then the areas of triangles AFH and HGE do not change. For convenience, allow points F and G to coincide with point B, as shown below:



Now, area of $\triangle ABH$ + area of $\triangle HBE$ = area of $\triangle ABE$

$$= \frac{1}{2}bh$$
$$= \frac{1}{2}(AE)(AB)$$
$$= \frac{1}{2}(15)(12)$$
$$= 90 \text{ (unshaded area)}$$

The shaded area = area of rectangle − area of $\triangle ABE$

$$= 12(30) - 90$$
$$= 360 - 90$$
$$= 270 \text{ square inches.}$$

Category 3 - NUMBER THEORY 1997
MEET #2

① The sum of four consecutive prime numbers is 168. What is the largest of these prime numbers?

② The GCF (Greatest Common Factor) of three numbers is 12. Two of those numbers are 36 and 72. What is the third number if it is between 125 and 150?

③ A Mersenne prime is a prime number which is the result of evaluating $2^N - 1$ where N is a prime number. What is the only Mersenne prime between 300 and 3000?

ANSWERS

① ______

② ______

③ ______

Number Theory SOLUTIONS - Meet #2

CATEGORY 3

1. 47
2. 132
3. 2047

① If the sum is 168, then their average must be close to 168 ÷ 4, or 42. Try a set of four consecutive primes near 42, and check their sum. 37 + 41 + 43 + 47 = 168, ∴ the largest of the primes is 47.

② The only multiples of 12 which are between 125 and 150 are 132 and 144. The number 144 can't be correct, because the GCF of 36, 72, and 144 is 36, not 12.
∴ 132 is the third number.

③ Experiment a bit with powers of 2:

N	2^N	$2^N - 1$
8	256	255
9	512	511
10	1024	1023
11	2048	2047 ← N is prime (11) and $2^N - 1$ is the only Mersenne prime between 300 and 3000.
12	4096	4095

① Find the fraction, in lowest terms, which is equal to the repeating decimal $0.2\overline{4}$.

② Simplify : $$\frac{(3.75)\left(1\frac{7}{9}\right)}{\frac{2}{3} - 0.25}$$

③ Find the fraction, in lowest terms, which is 60% greater than $\frac{5}{12}$.

ANSWERS

① ______

② ______

③ ______

CATEGORY 4

1. $\frac{11}{45}$
2. 16
3. $\frac{2}{3}$

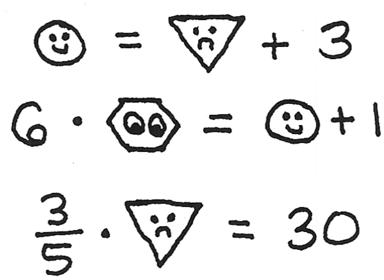
1. Let
$$\begin{aligned} 10x &= 2.44444\ldots \\ x &= 0.24444\ldots \\ \hline 9x &= 2.2 \end{aligned}$$
$$\frac{9x}{9} = \frac{2.2}{9}$$
$$x = \frac{2.2}{9} \cdot \frac{10}{10}$$
$$x = \frac{22}{90}, \text{ or } \frac{11}{45}$$

2. It may help to convert all numbers to a common form, such as fractions:
$$\frac{\left(\frac{15}{4}\right)\left(\frac{16}{9}\right)}{\frac{2}{3} - \frac{1}{4}} = \frac{\frac{\cancel{15}^{5}}{\cancel{4}_{1}} \cdot \frac{\cancel{16}^{4}}{\cancel{9}_{3}}}{\frac{8}{12} - \frac{3}{12}} = \frac{\frac{20}{3}}{\frac{5}{12}} = \frac{\cancel{20}^{4}}{\cancel{3}_{1}} \cdot \frac{\cancel{12}^{4}}{\cancel{5}_{1}} = 16$$

3. $$\frac{5}{12} + \frac{60}{100}\left(\frac{5}{12}\right) = \frac{5}{12} + \frac{\cancel{60}^{\cancel{3}^{1}}}{\cancel{100}_{\cancel{20}_{4}}} \cdot \frac{\cancel{5}^{1}}{\cancel{12}_{1}} = \frac{5}{12} + \frac{3}{12} = \frac{8}{12} = \frac{2}{3}$$

① Four consecutive multiples of 7 have a sum of 350. What is the product of the least and greatest of those four numbers?

②



Find the value of  .

③ Celcius and Fahrenheit temperatures are related by the following formula :

$$C = \frac{5}{9}(F - 32)$$

where C is the Celcius temperature, and F is the Fahrenheit temperature.
For what value is the Celcius temperature equal to the Fahrenheit temperature?

ANSWERS

① ______

② ______

③ ______

Algebra

CATEGORY 5

① 7546

② 112

③ −40

① x, $x+7$, $x+14$, $x+21$ } four consecutive multiples of 7

$$(x)+(x+7)+(x+14)+(x+21) = 350$$

$$4x+42 = 350$$

$$4x = 308$$

$$x = 77 \quad (\text{least})$$

$$x+21 = 98 \quad (\text{greatest}$$

∴ The product of the least and greatest is $77 \times 98 = 7546$.

② The last sentence offers the first decisive clue:

$$\tfrac{3}{5} \cdot ▽ = 30$$

$$\tfrac{5}{3} \cdot \tfrac{3}{5} \cdot ▽ = \tfrac{5}{3} \cdot 30$$

$$▽ = 50$$

Then ☺ = ▽ + 3 = 50 + 3 = 53

and

$$6 \cdot ⬡ = ☺ + 1$$

$$6 \cdot ⬡ = 53 + 1$$

$$6 \cdot ⬡ = 54$$

$$⬡ = 9$$

∴ ☺ + ▽ + ⬡ = 53 + 50 + 9 = 112.

③ Let C = F !

$$F = \tfrac{5}{9}(F - 32)$$

$$F = \tfrac{5}{9}F - \tfrac{5}{9} \cdot \tfrac{32}{1}$$

$$F - \tfrac{5}{9}F = \tfrac{5}{9}F - \tfrac{5}{9}F - \tfrac{160}{9}$$

$$\tfrac{4}{9}F = {}^{-}\tfrac{160}{9}$$

$$\tfrac{9}{4} \cdot \tfrac{4}{9}F = \tfrac{9}{4} \cdot {}^{-}\tfrac{160}{9}$$

$$F = {}^{-}40$$

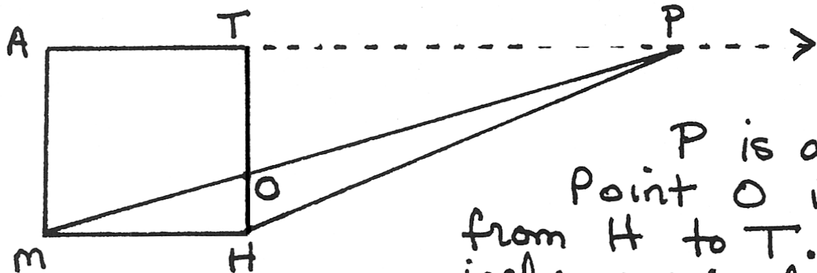
OR

$$C = \tfrac{5}{9}(C - 32)$$

solution follows the same as at the left

CATEGORY 6 - TEAM QUESTIONS 1997
MEET #2

① The average of a set of five numbers is 43. When a sixth number is added to the set, the average of the six numbers is 46. What is that sixth number?

② 

MATH is a square.
AM = 24 inches.
P is a point on line $\overleftrightarrow{AT}$.
Point O is 1/3 of the distance from H to T. How many square inches are in the area of triangle POH?

③ The sum of two positive integers is 13. What is the smallest possible sum of their reciprocals? Express your answer as a fraction in lowest terms.

④ If Brad gives you \$6, then you will both have the same amount of money. If you give Brad \$6 instead, he will have twice as much money as you. How many dollars does Brad have?

⑤ Each side of a scalene triangle has a length which is an even integer number of units. What is the least possible number of units in its perimeter?

⑥ Let the answers to questions #1-5 be represented by the letters A, B, C, D, E respectively. Evaluate the following expression:

$$\left(C\left[A-\left(\frac{3B}{E}-CD\right)\right]\right)^2$$

ANSWERS

① ______ = A

② ______ = B

③ ______ = C

④ ______ = D

⑤ ______ = E

⑥ ______

Team Questions

SOLUTIONS – Meet #2 1997

CATEGORY 6

1. 61
2. 192
3. $\frac{13}{42}$
4. 42
5. 18
6. 169

① If the average of five numbers is 43, then their sum is 43×5, or 215. If the average of six numbers is 46, then their sum is 46×6, or 276. The sixth number must be $276 - 215$, or 61.

② $\triangle MOH + \triangle POH = \triangle MPH$.
$\triangle MPH$ has the same base and altitude as the square. The area of the square $= bh = 24(24) = 576$ in². $\triangle MPH$ has half that area, or 288 in². $\triangle MOH$: area $= \frac{1}{2}bh = \frac{1}{2}(24)(8) = 96$ in². $\therefore$ area of $\triangle POH = 288 - 96 = 192$ in².

③ The smallest possible sum would be of two unit fractions with the greatest denominators, or $\frac{1}{6} + \frac{1}{7} = \frac{7}{42} + \frac{6}{42} = \frac{13}{42}$, which is already in lowest terms.

④

	current #	If Brad gives you \$6	If you give Brad \$6
You	x	$x+6$	$x-6$
Brad	y	$y-6$	$y+6$

$x+6=y-6$ $\qquad$ $2(x-6)=y+6$

$$\left.\begin{array}{l} x+6=y-6 \\ 2(x-6)=y+6 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x-y=-12 \\ 2x-12=y+6 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x-y=-12 \\ 2x-y=18 \end{array}\right\} \text{subtr.}$$

$$-x = -30$$
$$x = 30$$
$$y = 42$$

$\therefore$ Brad has \$42.

⑤ The smallest set of different even positive integers, is 2, 4, 6. However, these could not be sides of a triangle, as $2+4=6$. The sum of the lengths of any two sides of a triangle must always be greater than the length of the third side. $\therefore$ the three sides are 4, 6, 8, as $4+6>8$.
$4+6+8=18$.

⑥ $$\left(\frac{13}{42}\left[61-\left(\frac{3\cdot 192}{18}-\frac{13}{42}\cdot 42\right)\right]\right)^2$$
$$=\left(\frac{13}{42}\left[61-(32-13)\right]\right)^2$$
$$=\left(\frac{13}{42}\left[61-19\right]\right)^2$$
$$=\left(\frac{13}{42}\left[42\right]\right)^2$$
$$=(13)^2$$
$$=169$$